

# SOBOLEV REGULARITY OF SOLUTIONS OF THE COHOMOLOGICAL EQUATION

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## 1. INTRODUCTION

We prove the sharpest results available on the loss of regularity for solutions of the cohomological equation for translation flows. For *any given translation surface* and for the directional flow in almost all directions the

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smallest loss Sobolev regularity available to the Fourier analysis methods developed in [For97] is essentially  $3 + \epsilon$  (for any  $\epsilon > 0$ ). We remark that this is the best result available for the flow of rational polygonal billiards in almost all directions. The motivation for improving the estimate on the loss of regularity obtained in [For97] was provided by a question of Marmi, Moussa and Yoccoz [MMY05]. We also remark that their results, on the related problem of solutions of the cohomological equation for almost all interval exchange transformations (IET's), do not apply to rational billiards for the well-known reason that IET's induced by rational billiard flows form a zero measure set in the space of all IET's.

For *almost all translation surfaces* in every stratum of the moduli space, we prove the refined, optimal result that the loss of Sobolev regularity for the directional flow in almost all directions is  $1 + \epsilon$  (for any  $\epsilon > 0$ ). In this case, in fact we prove that for any function of Sobolev order  $s > 1$ , the solution and its derivatives up to order  $k < s - 1$  are  $L^\infty$  functions on the surface. We also determine precisely the Sobolev orders of the distributional obstructions to the existence of solutions (first constructed in [For97]) in terms of the Kontsevich-Zorich Lyapunov exponents [Zor96], [Kon97]. As a consequence we are able to determine the exact codimension of coboundaries for every Sobolev regularity class of the transfer function. For instance, the codimension of coboundaries with square-integrable transfer functions (in the space of functions of Sobolev order  $s > 1$ ) is exactly equal to the genus of the surface. For such coboundaries the transfer function is actually in  $L^\infty$ .

These results implies quite immediately corresponding results for interval exchange transformations, which improve on the loss of regularity established in [MMY05]. We should point out that in that paper the authors are mostly concerned with *Diophantine conditions* on interval exchange transformations for which the cohomological equation admits smooth solution, while we have not investigated this question at all. The reason is that the full measure sets of systems for which our results (as well as [MMY05]) are determined by several conditions which always include *Oseledec regularity* (or rather a weaker *coherence* property) with respect to Kontsevich-Zorich renormalization cocycle. The Oseledec's theorem is invoked to ensure that the set of regular (coherent) IET's has full measure. Any substantial progress over [MMY05] would have to succeed in characterizing explicitly a full measure set of regular (coherent) points without relying on the Oseledec's theorem. To the best of our knowledge this goal is still beyond reach.

There are several motivations for this work. The study of cohomological equations is a relevant part of the theory of (smooth) dynamical systems directly connected to basic questions such as *triviality of time-changes* for

flows, *asymptotic of ergodic averages* and the *smooth conjugacy problem* via linearization and Nash-Moser implicit function theorem. In the hyperbolic case (for dynamical systems with exponential divergence of nearby orbits) such there are extensive, deep results on the cohomological equation pioneered in the work of Livsic [Liv71], later developed by several authors (see [GK80], [CEG84], [dlMM86]). The completely different case of (Diophantine) linear flows on the torus is well-known, since the cohomological equation for such systems is closely related to the linearized equation in the classical KAM theory for Hamiltonian flows. This in an example of elliptic dynamics (no divergence of nearby orbits) which can be studied to a great extent by the classical theory of Fourier series. It is characterized by the ‘small divisors’ appearing in the Fourier coefficients of solutions, which lead to a loss of regularity. It is not difficult to see that the optimal loss of Sobolev regularity for the full measure set of Roth flows is  $1 + \epsilon$  and that for such flows any zero average function of Sobolev order  $s > 1$  is an  $L^2$  coboundary. It can be proved by our methods (and by the Gottschalk-Hedlund theorem [GH55]) that the transfer function is in fact continuous. We have not been able to locate this result in the literature, however it is well within reach of the methods of [Her83], Chap. VI, §3. However, only the measure zero case of rotation numbers of *constant type* seems to have been explicitly considered there.

For systems with intermediate behavior, that is, for elliptic systems with singularities or for parabolic systems (characterized by polynomial divergence of nearby orbits) much less is known. The author discovered in [For97] that the cohomological equation for generic translation flows (or equivalently for generic IET’s) has finitely smooth solutions for sufficiently smooth data under finitely many distributional conditions. In other terms, on one hand the problem shares a typical feature of ‘small divisors’ problems, namely the finite loss of regularity of solutions with respect to the data; on the other hand, a new phenomenon appears: the existence of infinitely many independent distributional obstructions (of increasing order) which are not given by invariant measures. In [For97] only a rough estimate for the loss of derivatives is explicitly obtained ( $\leq 9$ ). Our goal in this paper is to improve such estimate as much as possible. In joint papers with L. Flaminio the authors have investigated the existence of smooth solutions of the cohomological equation for horocycle flows (on surfaces constant negative curvature) [FF03], for generic nilflows on quotients of the Heisenberg group [FF06] and generic nilflows on general nilmanifolds [FF07]. In all cases the fundamental features of finite loss derivatives and of the existence of infinitely many independent distributional obstructions have been established (although the structure of the space of invariant distributions is

significantly different for IET's, horocycle flows and nilflows). For horocycle flows and for Heisenberg nilflows it was possible to estimate that the loss of Sobolev regularity is  $1 + \epsilon$  (for any  $\epsilon > 0$ ) and to establish the conjectural relation that the Sobolev order of the distributional obstructions be related to the Lyapunov exponents of the distribution under the appropriate renormalization dynamics. In this paper we prove analogous results for generic translation flows. We should point out that for generic nilflows on general nilmanifolds the loss of regularity and the regularity of the distributional obstructions seem to depend on the depth and rank of the nilpotent group considered, although no lower bound was established in [FF07].

Let  $q$  be a *holomorphic orientable quadratic differential* on a Riemann surface  $M$  of genus  $g \geq 1$ . The horizontal and vertical measured foliations (in the Thurston's sense) associated to a holomorphic quadratic differential  $q$  on  $M$  are defined as  $\mathcal{F}_q = \{\Im(q^{1/2}) = 0\}$  (the horizontal foliation) and  $\mathcal{F}_q = \{\Re(q^{1/2}) = 0\}$ . Such foliations are well-defined even in the case that there is no globally defined square root of the quadratic differential. The horizontal foliation is endowed with the transverse measure given by  $|\Im(q^{1/2})|$ , the vertical foliation is endowed with the transverse measure given by  $|\Re(q^{1/2})|$ . The quadratic differential is called *orientable* if the horizontal and vertical foliations are both orientable. Orientability is equivalent to the condition that the quadratic differential is globally the square of a holomorphic (abelian) differential. The structure induced by an orientable holomorphic quadratic differential (or by a holomorphic abelian differential) can also be described as follows. There is a flat metric  $R_q$  associated with any quadratic differential  $q$  on  $M$ . Such a metric has conical singularities as the finite set  $\Sigma_q = \{p \in M | q(p) = 0\}$ . If  $q$  is orientable there exists a (positively oriented) parallel orthonormal frame  $\{S_q, T_q\}$  of the tangent bundle  $TM|_{M \setminus \Sigma_q}$  such that  $S_q$  is tangent to the horizontal foliation  $\mathcal{F}_q$  and  $T_q$  is tangent to the vertical foliation  $\mathcal{F}_{-q}$  everywhere on  $M \setminus \Sigma_q$ . In other terms, *the flat metric  $R_q$  has trivial holonomy*. In another equivalent formulation, any orientable holomorphic quadratic differential determines a *translation structure* on  $M$ , that is, an equivalence class of atlases with transition functions given by translation of the euclidean plane (see for instance the excellent survey [MT05], §1.8. For a given orientable quadratic differential  $q$  on a Riemann surface  $M$ , we will consider the one-parameter family of vector fields on  $M \setminus \Sigma_q$  defined as

$$(1.1) \quad S_\theta := \cos \theta S_q + \sin \theta T_q, \quad \theta \in S^1.$$

The vector field  $S_\theta$  is a parallel normalized vector field in the direction at angle  $\theta \in S^1$  with the horizontal. We remark that it is not defined as the singular set  $\Sigma_q$  of the flat metric. hence the flow it generates is defined (almost everywhere) on the complement of the union of all separatrices

of the orbit foliation (a measure foliation). The singularities of the orbit foliation are all saddle-like, but the saddles are degenerate if the order of zero of the quadratic differential at the singularity is strictly greater than 2. In fact, since the quadratic differential is supposed to be orientable it has zeroes of even order and the orbit foliations of the vector fields (1.1) has  $m$  stable and  $m$  unstable separatrices at any zero of order  $2m$ .

Our goal is to investigate the loss of (Sobolev) regularity of solutions of the *cohomological equation*  $S_\theta u = f$  for Lebesgue almost all  $\theta \in S^1$ . The author proved in [For97] that if the function  $f$  is sufficiently regular, satisfies a finite number of independent distributional conditions (which include conditions on the jets at the singularities) then there exists a finitely smooth solution (unique up to additive constants). The loss of regularity was estimated in that paper to be no more than 9 derivatives in the Sobolev sense.

If  $q$  is any orientable quadratic differential, the regularity of functions on the translation surface  $(M, q)$  is expressed in terms of a family  $\{H_q^s(M) | s \in \mathbb{R}\}$  of *weighted Sobolev spaces*. Such spaces were introduced in [For97] for all  $s \in \mathbb{Z}$  as follows. Let  $\omega_q$  be the standard (degenerate) volume form on  $M$  of the flat metric  $R_q$ . The space  $H_q^0(M)$  is the space  $L^2(M, \omega_q)$  of square-integrable functions. For  $k \in \mathbb{N}$ , the space  $H_q^k(M)$  is the subspace of functions  $f \in H_q^0(M)$  such that the weak derivatives  $S_q^i T_q^j f \in H_q^0(M)$  [and  $T_q^i S_q^j f \in H_q^0(M)$ ] for all  $i + j \leq k$  and the space  $H_q^{-k}(M)$  is the dual Hilbert space  $H_q^k(M)^*$ . In §2 of this paper we introduce *weighted Sobolev spaces with arbitrary (real) exponents* by methods of interpolation theory. Although the Sobolev norms we construct do not form an interpolation family in the sense of (holomorphic) interpolation theory, they do satisfy a standard interpolation inequality. The weighted Sobolev spaces combine standard Sobolev smoothness conditions on  $M \setminus \Sigma_q$  with restrictions on the jet of the functions at the singular set  $\Sigma_q \subset M$ .

As discovered in [For97], for functions  $f \in C_0^\infty(M \setminus \Sigma_q)$  the space of all distributional obstructions to the existence of a solution  $u \in C_0^\infty(M \setminus \Sigma_q)$  of the cohomological equation  $S_\theta u = f$  coincides for almost all  $\theta \in S^1$  with the infinite dimensional space of all  $S_\theta$ -invariant distributions:

$$(1.2) \quad \mathcal{J}_{q,\theta}(M \setminus \Sigma_q) := \{\mathcal{D} \in \mathcal{D}'(M \setminus \Sigma_q) \mid S_\theta \mathcal{D} = 0 \text{ in } \mathcal{D}'(M \setminus \Sigma_q)\}.$$

For data  $f \in H_q^s(M)$  of finite Sobolev differentiability, a complete set of obstructions is given for almost all  $\theta \in S^1$  by the finite dimensional subspace of invariant distributions

$$(1.3) \quad \mathcal{J}_{q,\theta}^s(M) := \{\mathcal{D} \in H_q^{-s}(M) \mid S_\theta \mathcal{D} = 0 \text{ in } H_q^{-s}(M)\}.$$

The goal of this paper is to prove *optimal* estimates on the Sobolev regularity of solutions of the cohomological equation  $S_\theta u = f$  and on the

dimension of the spaces  $\mathcal{J}_{q,\theta}^s(M)$  of invariant distributions for all  $s > 0$ . In §5.1, Theorem 5.1, we prove

**Theorem A1.** *Let  $q$  be any orientable holomorphic quadratic differential. Let  $k \in \mathbb{N}$  be any integer such that  $k \geq 3$  and let  $s > k$  and  $r < k - 3$ . For almost all  $\theta \in S^1$  (with respect to the Lebesgue measure), there exists a constant  $C_{r,s}(\theta) > 0$  such that the following holds. If  $f \in H_q^s(M)$  is such that  $\mathcal{D}(f) = 0$  for all  $\mathcal{D} \in \mathcal{J}_{q,\theta}^s(M)$ , the cohomological equation  $S_\theta u = f$  has a solution  $u \in H_q^r(M)$  satisfying the following estimate:*

$$(1.4) \quad |u|_r \leq C_{r,s}(\theta) |f|_s .$$

The dimensions of the spaces of invariant distributions can be estimated as follows (see §3.3, Corollary 3.20 and Theorem 3.21):

**Theorem A2.** *Let  $q$  be any orientable holomorphic quadratic differential. Let  $k \in \mathbb{N}$  be any integer such that  $k \geq 3$  and let  $k < s \leq k + 1$ . For almost all  $\theta \in S^1$  (with respect to the Lebesgue measure),*

$$(1.5) \quad 1 + 2(k - 2)(g - 1) \leq \dim \mathcal{J}_{q,\theta}^s(M) \leq 1 + (2k - 1)(g - 1) .$$

The proof of the above results is essentially based on the harmonic analysis methods developed in [For97]. We remark that no other methods are known for the case of an *arbitrary* orientable quadratic differential.

We prove much sharper results for *almost all* orientable quadratic differentials. The moduli space of orientable holomorphic quadratic differentials  $q$  on some Riemann surface  $M_q$  with a given pattern of zeroes, that is, with zeroes of (even) multiplicities  $\kappa = (k_1, \dots, k_\sigma)$  at a finite set  $\Sigma_q = \{p_1, \dots, p_\sigma\} \subset M_q$  is a stratum  $\mathcal{M}_\kappa$  of the moduli space  $\mathcal{M}_g$  of all holomorphic quadratic differential. Let  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  be the subsets of quadratic differential of total area equal to 1. It was proved by H. Masur [Mas82] and W. Veech [Vee86] that each stratum  $\mathcal{M}_\kappa^{(1)}$  carries an absolutely continuous probability measure  $\mu_\kappa^{(1)}$ , invariant under the action of the Teichmüller geodesic flow, which is ergodic when restricted to each connected component of  $\mathcal{M}_\kappa^{(1)}$  (the connected components of strata of orientable quadratic differentials were classified in [KZ03]). In fact, there is natural action of the group  $SL(2, \mathbb{R})$  on the moduli space  $\mathcal{M}_g^{(1)}$  such that the Teichmüller geodesic flow corresponds to the action of the diagonal subgroup of  $SL(2, \mathbb{R})$  and the measure  $\mu_\kappa^{(1)}$  is  $SL(2, \mathbb{R})$  invariant.

In [Kon97] M. Kontsevich introduced a renormalization cocycle for translation flows, inspired to the Rauzy-Veech-Zorich cocycle for interval exchange transformations. The Kontsevich-Zorich cocycle is a dynamical system on an orbifold vector bundle over  $\mathcal{M}_\kappa^{(1)}$  with fiber the first cohomology  $H^1(M_q, \mathbb{R})$  of the Riemann surface carrying the orientable holomorphic

quadratic differential  $q \in \mathcal{M}_\kappa^{(1)}$ . The action of such a dynamical system is (by definition of a cocycle) linear on the fibers and projects onto the Teichmüller geodesic flow on the base  $\mathcal{M}_\kappa^{(1)}$ . Since the cocycle is symplectic, for any probability measure  $\mu$  on a stratum  $\mathcal{M}_\kappa^{(1)}$ , the *Lyapunov spectrum* of the Kontsevich-Zorich cocycle takes the form:

$$(1.6) \quad \lambda_1^\mu \geq \dots \geq \lambda_g^\mu \geq 0 \geq \lambda_{g+1}^\mu = -\lambda_g^\mu \geq \dots \geq \lambda_{2g}^\mu = -\lambda_1^\mu.$$

In addition, it is not difficult to prove that  $\lambda_1^\mu = 1$ . A probability measure  $\mu$  on a stratum  $\mathcal{M}_\kappa^{(1)}$ , invariant under the Teichmüller geodesic flow, will be called (a) *SO(2, ℝ)-absolutely continuous* if it induces absolutely continuous measures on every orbit of the circle group  $SO(2, \mathbb{R}) \subset SL(2, \mathbb{R})$ ; (b) *KZ-hyperbolic* if all the Lyapunov exponents in (1.6) are non-zero. It is immediate that all  $SL(2, \mathbb{R})$ -invariant measures are  $SO(2, \mathbb{R})$ -absolutely continuous. It was first proved in [For02] that the measure  $\mu_\kappa^{(1)}$  is KZ-hyperbolic. A different proof that also reaches the stronger conclusion that the exponents (1.6) are all distinct has been given more recently by A. Avila and M. Viana [AV05] who have thus completed the proof of the Zorich-Kontsevich conjectures [Zor96], [Kon97] on the Lyapunov spectrum of the Kontsevich-Zorich cocycle (and its discrete counterparts).

Our sharpest results are proved for *almost all* quadratic differentials with respect to any  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic, Teichmüller invariant, probability measure on any stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. The smoothness informations on the solutions is stronger than just Sobolev  $L^2$  regularity and it is naturally encoded by the following spaces. For any  $k \in \mathbb{N}$ , let  $B_q^k(M)$  be the space of all functions  $u \in H_q^k(M)$  such that  $S_q^i T_q^j u = T_q^i S_q^j u \in L^\infty(M)$  for all pairs of integers  $(i, j)$  such that  $0 \leq i + j \leq k$ . The space  $B_q^k(M)$  is endowed with the norm defined as follows: for any  $u \in B_q^k(M)$ ,

$$(1.7) \quad |u|_{k,\infty} := \left[ \sum_{i+j \leq k} |S_q^i T_q^j u|_\infty^2 \right]^{1/2} = \left[ \sum_{i+j \leq k} |T_q^i S_q^j u|_\infty^2 \right]^{1/2}.$$

For  $s \in [k, k+1)$ , let  $B_q^s(M) := B_q^k(M) \cap H_q^s(M)$  endowed with the norm defined as follows: for any  $u \in B_q^s(M)$ ,

$$(1.8) \quad |u|_{s,\infty} := (|u|_{k,\infty}^2 + |u|_s^2)^{1/2}.$$

In §5.3, Theorem 5.19, we prove the following:

**Theorem B1.** *Let  $\mu$  be any  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic probability measure on any stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. Let  $s > 1$  and let  $r < s - 1$ . For  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for*

almost all  $\theta \in S^1$  (with respect to the Lebesgue measure), there exists a constant  $C_{r,s}(\theta) > 0$  such that the following holds. If  $f \in H_q^s(M)$  is such that  $\mathcal{D}(f) = 0$  for all  $\mathcal{D} \in \mathcal{J}_{q,\theta}^s(M)$ , the cohomological equation  $S_\theta u = f$  has a solution  $u \in B_q^r(M)$  satisfying the following estimate:

$$(1.9) \quad |u|_{r,\infty} \leq C_{r,s}(\theta) |f|_s .$$

The regularity of invariant distributions can be precisely determined as follows (see §4.3, Corollary 4.33, and §5.3, Theorem 5.19). For any  $q \in \mathcal{M}_\kappa^{(1)}$  and any distribution  $\mathcal{D} \in \mathcal{D}'(M \setminus \Sigma_q)$ , the *weighted Sobolev order* is the number

$$(1.10) \quad \mathcal{O}_q^H(\mathcal{D}) = \inf\{s \in \mathbb{R}^+ | \mathcal{D} \in H_q^{-s}(M)\} .$$

Let  $\mathcal{J}_{q,\theta}(M) := \cup\{\mathcal{J}_{q,\theta}^s(M) | s \geq 0\}$  denote the space of all  $S_\theta$ -invariant distribution of finite Sobolev order and let  $\hat{\mathcal{J}}_{q,\theta}(M) \subset \mathcal{J}_{q,\theta}(M)$  be the subspace of invariant distributions vanishing on constant functions. It follows immediately by the definitions that  $\mathcal{J}_{q,\theta}(M) = \mathbb{C} \oplus \hat{\mathcal{J}}_{q,\theta}(M)$ .

**Theorem B2.** *Let  $\mu$  be any  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic probability measure on any stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. For  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for almost all  $\theta \in S^1$  (with respect to the Lebesgue measure), the space  $\hat{\mathcal{J}}_{q,\theta}(M)$  has a basis  $\{\mathcal{D}_{i,j}(\theta)\}$  such that*

$$(1.11) \quad \mathcal{O}_q^H(\mathcal{D}_{i,j}(\theta)) = \lambda_i^\mu - (j+1), \quad i \in \{2, \dots, 2g-1\}, j \in \mathbb{N} \cup \{0\} .$$

*In addition, the basis  $\{\mathcal{D}_{i,j}(\theta)\}$  can be generated from the finite dimensional subsystem  $\{\mathcal{D}_2, \dots, \mathcal{D}_{2g-1}\}$  by the following differential relations:*

$$(1.12) \quad \mathcal{D}_{i,j}(\theta) = T_\theta^j \mathcal{D}_{i,0}(\theta), \quad i \in \{2, \dots, 2g-1\}, j \in \mathbb{N} \cup \{0\} .$$

The above Theorems B1 and B2 are proved by methods based on renormalization, which were inspired by the work of Marmi-Moussa-Yoccoz [MMY05]. However, our approach differs from theirs since we explicitly assume that the Lyapunov exponents of the Kontsevich-Zorich cocycle are all non-zero. The main idea of the argument, as in [MMY05], is to prove uniform estimates for ergodic integrals of weakly differentiable functions, then apply a version of Gottshalk-Hedlund theorem. The asymptotics of ergodic averages of functions in  $H^1(M)$  was studied by the author in [For02], where the Kontsevich-Zorich conjectures on the deviation of ergodic averages for smooth functions (formulated in [Kon97]) were proved. The approach of [For02] is based on the analysis of distributional cocycles over the Teichmüller flow which extend the Kontsevich-Zorich cocycles. The estimates proved in [For02] are (barely) not strong enough to yield the required uniform boundedness of ergodic averages under the appropriate distributional conditions. In §4.2 of this paper we have recalled the definition of



distributional cocycles, and we have strengthened the estimates proved in [For02] under the slightly stronger (and correct) assumption that the functions considered belong to  $H_q^s(M)$  for some  $s > 1$ .

Another important technical issue that separates Theorems B1 and B2 from the less precise Theorems A1 and A2 is related to interpolation theory in the presence of distributional obstructions. In the general case, we have not been able to overcome the related difficulties, hence the lack of precision of Theorem A1 and A2 for intermediate Sobolev regularity. In the *generic* case of Theorems B1 and B2 we have been able to prove a remarkable linear independence property of invariant distributions which makes interpolation possible in the construction of solutions of the cohomological equation.

We introduce the following definition (see Definition 5.11). A finite system  $\{\mathcal{D}_1, \dots, \mathcal{D}_J\} \subset H_q^{-\sigma}(M)$  of finite order distributions is called  $\sigma$ -regular (with respect to the family  $\{H_q^s(M)\}$  of weighted Sobolev spaces) if for any  $\tau \in (0, 1]$  there exists a dual system  $\{u_1(\tau), \dots, u_J(\tau)\} \subset H_q^\sigma(M)$  (that is, the identities  $\mathcal{D}_i(u_j(\tau)) = \delta_{ij}$  hold for all  $i, j \in \{1, \dots, J\}$  and all  $\tau \in (0, 1]$ ) such that the following estimates hold. For all  $0 \leq r \leq \sigma$  and all  $\epsilon > 0$ , there exists a constant  $C_r^\sigma(\epsilon) > 0$  such that, for all  $i, j \in \{1, \dots, J\}$ ,

$$(1.13) \quad |u_j(\tau)|_r \leq C_r^\sigma(\epsilon) \tau^{\mathcal{O}^H(\mathcal{D}_j) - r - \epsilon}.$$

A finite system  $\{\mathcal{D}_1, \dots, \mathcal{D}_J\} \subset H_q^{-s}(M)$  of finite order distributions will be called *regular* if it is  $\sigma$ -regular for any  $\sigma \geq s$ . A finite dimensional subspace  $\mathcal{J} \subset H_q^{-s}(M)$  of finite order distributions will be called  $\sigma$ -regular [regular] if it admits a  $\sigma$ -regular [regular] basis.

We have proved that the spaces of distributional obstructions for the cohomological equation are regular in the above sense (see Theorem 5.18).

**Theorem C.** *Let  $\mu$  be any  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic probability measure on any stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. For  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , for almost all  $\theta \in S^1$  and for all  $s > 0$ , the space  $\mathcal{J}_{q,\theta}^s(M) \subset H_q^{-s}(M)$  of  $S_\theta$ -invariant distributions is regular.*

## 2. FRACTIONAL WEIGHTED SOBOLEV SPACES

In [For97] we have introduced a natural scale of weighted Sobolev spaces with integer exponent associated with any orientable holomorphic quadratic differential  $q$  on a Riemann surface  $M$  (of genus  $g \geq 2$ ). In this section we extend the definition of weighted Sobolev spaces to arbitrary (real) exponents by methods of interpolation theory.

**2.1. Weighted Sobolev spaces.** Let  $\Sigma_q := \{p_1, \dots, p_\sigma\} \subset M$  be the set of zeros of the holomorphic quadratic differential  $q$ , of even orders  $(k_1, \dots, k_\sigma)$  respectively with  $k_1 + \dots + k_\sigma = 4g - 4$ . Let  $R_q := |q|^{1/2}$  be

the flat metric with cone singularities at  $\Sigma_q$  induced by the quadratic differential  $q$  on  $M$ . With respect to a holomorphic local coordinate  $z = x + iy$ , the quadratic differential  $q$  has the form  $q = \phi(z)dz^2$ , where  $\phi$  is a locally defined holomorphic function, and, consequently,

$$(2.1) \quad R_q = |\phi(z)|^{1/2}(dx^2 + dy^2)^{1/2}, \quad \omega_q = |\phi(z)| dx \wedge dy.$$

The metric  $R_q$  is flat, it is degenerate at the finite set  $\Sigma_q$  of zeroes of  $q$  and, if  $q$  is orientable, it has trivial holonomy, hence  $q$  induces a structure of *translation surface* on  $M$ .

The weighted  $L^2$  space is the standard space  $L_q^2(M) := L^2(M, \omega_q)$  with respect to the area element  $\omega_q$  of the metric  $R_q$ . Hence the weighted  $L^2$  norm  $|\cdot|_0$  are induced by the hermitian product  $\langle \cdot, \cdot \rangle_q$  defined as follows: for all functions  $u, v \in L_q^2(M)$ ,

$$(2.2) \quad \langle u, v \rangle_q := \int_M u \bar{v} \omega_q.$$

Let  $\mathcal{F}_q$  be the *horizontal foliation*,  $\mathcal{F}_{-q}$  be the *vertical foliation* for the holomorphic quadratic differential  $q$  on  $M$ . The foliations  $\mathcal{F}_q$  and  $\mathcal{F}_{-q}$  are measured foliations (in the Thurston's sense):  $\mathcal{F}_q$  is the foliation given (locally) by the equation  $\Im(q^{1/2}) = 0$  endowed with the invariant transverse measure  $|\Im(q^{1/2})|$ ,  $\mathcal{F}_{-q}$  is the foliation given (locally) by the equation  $\Re(q^{1/2}) = 0$  endowed with the invariant transverse measure  $|\Re(q^{1/2})|$ . If the quadratic differential  $q$  is orientable, since the metric  $R_q$  is flat with trivial holonomy, there exist commuting vector fields  $S_q$  and  $T_q$  on  $M \setminus \Sigma_q$  such that

- (1) The frame  $\{S_q, T_q\}$  is a parallel orthonormal frame with respect to the metric  $R_q$  for the restriction of the tangent bundle  $TM$  to the complement  $M \setminus \Sigma_q$  of the set of cone points;
- (2) the vector field  $S_q$  is tangent to the horizontal foliation  $\mathcal{F}_q$ , the vector field  $T_q$  is tangent to the vertical foliation  $\mathcal{F}_{-q}$  on  $M \setminus \Sigma_q$  [For97].

In the following we will often drop the dependence of the vector fields  $S_q$ ,  $T_q$  on the quadratic differential in order to simplify the notations. We have:

- (1)  $\mathcal{L}_{S_q}\omega_q = \mathcal{L}_{T_q}\omega_q = 0$  on  $M \setminus \Sigma_q$ , that is, the area form  $\omega_q$  is invariant with respect to the flows generated by  $S$  and  $T$ ;
- (2)  $\iota_{S_q}\omega_q = \Re(q^{1/2})$  and  $\iota_{T_q}\omega_q = \Im(q^{1/2})$ , hence the 1-forms  $\eta_S := \iota_S\omega_q$ ,  $\eta_T := -\iota_T\omega_q$  are smooth and closed on  $M$  and  $\omega_q = \eta_T \wedge \eta_S$ .

It follows from the area-preserving property (1) that the vector field  $S$ ,  $T$  are anti-symmetric as densely defined operators on  $L_q^2(M)$ , that is, for all functions  $u, v \in C_0^\infty(M \setminus \Sigma_q)$ , (see [For97], (2.5)),

$$(2.3) \quad \langle Su, v \rangle_q = -\langle u, Sv \rangle_q, \quad \text{respectively} \quad \langle Tu, v \rangle_q = -\langle u, Tv \rangle_q.$$

In fact, by Nelson's criterion [Nel59], Lemma 3.10, the anti-symmetric operators  $S, T$  are *essentially skew-adjoint* on the Hilbert space  $L_q^2(M)$ .

The *weighted Sobolev norms*  $|\cdot|_k$ , with integer exponent  $k > 0$ , are the euclidean norms, introduced in [For97], induced by the hermitian product defined as follows: for all functions  $u, v \in L_q^2(M)$ ,

$$(2.4) \quad \langle u, v \rangle_k := \frac{1}{2} \sum_{i+j \leq k} \langle S^i T^j u, S^i T^j v \rangle_q + \langle T^i S^j u, T^i S^j v \rangle_q.$$

The *weighted Sobolev norms* with integer exponent  $-k < 0$  are defined to be the dual norms.

The *weighted Sobolev space*  $H_q^k(M)$ , with integer exponent  $k \in \mathbb{Z}$ , is the Hilbert space obtained as the completion with respect to the norm  $|\cdot|_k$  of the maximal *common invariant domain*

$$(2.5) \quad H_q^\infty(M) := \bigcap_{i,j \in \mathbb{N}} D(\bar{S}^i \bar{T}^j) \cap D(\bar{T}^i \bar{S}^j).$$

of the closures  $\bar{S}, \bar{T}$  of the essentially skew-adjoint operators  $S, T$  on  $L_q^2(M)$ . The weighted Sobolev space  $H_q^{-k}(M)$  is isomorphic to the dual space of the Hilbert space  $H_q^k(M)$ , for all  $k \in \mathbb{Z}$ .

Since the vector fields  $S, T$  commute (infinitesimally) on  $M \setminus \Sigma_q$ , the following weak commutation identity holds on  $M$ .

**Lemma 2.1.** ([For97], Lemma 3.1) *For all functions  $u, v \in H_q^1(M)$ ,*

$$(2.6) \quad \langle Su, Tv \rangle_q = \langle Tu, Sv \rangle_q.$$

By the anti-symmetry property (2.3) and the commutativity property (2.6), the frame  $\{S, T\}$  yields an essentially skew-adjoint action of the Lie algebra  $\mathbb{R}^2$  on the Hilbert space  $L_q^2(M)$  with common domain  $H_q^1(M)$ . If  $\Sigma_q \neq \emptyset$ , the (flat) Riemannian manifold  $(M \setminus \Sigma_q, R_q)$  is not complete, hence its Laplacian  $\Delta_q$  is not essentially self-adjoint on  $C_0^\infty(M \setminus \Sigma_q)$ . By a theorem of Nelson [Nel59], §9, this is equivalent to the non-integrability of the action of  $\mathbb{R}^2$  as a Lie algebra (to an action of  $\mathbb{R}^2$  as a Lie group).

Following [For97], the Fourier analysis on the flat surface  $M_q$  will be based on a canonical self-adjoint extension  $\Delta_q^F$  of the Laplacian  $\Delta_q$ , called the *Friedrichs extension*, which is uniquely determined by the *Dirichlet hermitian form*  $\mathcal{Q} : H_q^1(M) \times H_q^1(M) \rightarrow \mathbb{C}$ . We recall that, for all  $u, v \in H_q^1(M)$ ,

$$(2.7) \quad \mathcal{Q}(u, v) := \langle Su, Sv \rangle_q + \langle Tu, Tv \rangle_q.$$

**Theorem 2.2.** ([For97], Th. 2.3) *The hermitian form  $\mathcal{Q}$  on  $L_q^2(M)$  has the following spectral properties:*

- (1)  $\mathcal{Q}$  is positive semi-definite and the set  $\text{EV}(\mathcal{Q})$  of its eigenvalues is a discrete subset of  $[0, +\infty)$ ;
- (2) Each eigenvalue has finite multiplicity, in particular  $0 \in \text{EV}(\mathcal{Q})$  is simple and the kernel of  $\mathcal{Q}$  consists only of constant functions;
- (3) The space  $L_q^2(M)$  splits as the orthogonal sum of the eigenspaces. In addition, all eigenfunctions are  $C^\infty$  (real analytic) on  $M$ .

The Weyl asymptotics holds for the eigenvalue spectrum of the Dirichlet form. For any  $\Lambda > 0$ , let  $N_q(\Lambda) := \text{card}\{\lambda \in \text{EV}(\mathcal{Q}) / \lambda \leq \Lambda\}$ , where each eigenvalue  $\lambda \in \text{EV}(\mathcal{Q})$  is counted according to its multiplicity.

**Theorem 2.3.** ([For97], Th. 2.5) *There exists a constant  $C > 0$  such that*

$$(2.8) \quad \lim_{\Lambda \rightarrow +\infty} \frac{N_q(\Lambda)}{\Lambda} = \text{vol}(M, R_q) .$$

Let  $\partial_q^\pm := S_q \pm i T_q$  be the *Cauchy-Riemann operators* induced by the holomorphic orientable quadratic differential  $q$  on  $M$ , introduced in [For97], §3. Let  $\mathcal{M}_q^\pm \subset L_q^2(M)$  be the subspaces of meromorphic, respectively anti-meromorphic functions (with poles at  $\Sigma_q$ ). By the Riemann-Roch theorem, the subspaces  $\mathcal{M}_q^\pm$  have the same complex dimension equal to the genus  $g \geq 1$  of the Riemann surface  $M$ . In addition,  $\mathcal{M}_q^+ \cap \mathcal{M}_q^- = \mathbb{C}$ , hence

$$(2.9) \quad H_q := (\mathcal{M}_q^+)^{\perp} \oplus (\mathcal{M}_q^-)^{\perp} = \{u \in L_q^2(M) \mid \int_M u \omega_q = 0\} .$$

Let  $H_q^1 := H_q \cap H_q^1(M)$ . By Theorem 2.2, the restriction of the hermitian form to  $H_q^1$  is positive definite, hence it induces a norm. By the Poincaré inequality (see [For97], Lemma 2.2 or [For02], Lemma 6.9), the Hilbert space  $(H_q^1, \mathcal{Q})$  is isomorphic to the Hilbert space  $(H_q^1, \langle \cdot, \cdot \rangle_1)$ .

**Proposition 2.4.** ([For97], Prop. 3.2) *The Cauchy-Riemann operators  $\partial_q^\pm$  are closable operators on the common domain  $C_0^\infty(M \setminus \Sigma_q) \subset L_q^2(M)$  and their closures (denote by the same symbols) have the following properties:*

- (1) the domains  $D(\partial_q^\pm) = H_q^1(M)$  and the kernels  $N(\partial_q^\pm) = \mathbb{C}$ ;
- (2) the ranges  $R_q^\pm := \text{Ran}(\partial_q^\pm) = (\mathcal{M}_q^\mp)^{\perp}$  are closed in  $L_q^2(M)$ ;
- (3) the operators  $\partial_q^\pm : (H_q^1, \mathcal{Q}) \rightarrow (R_q^\pm, \langle \cdot, \cdot \rangle_q)$  are isometric.

Let  $\mathcal{E} = \{e_n \mid n \in \mathbb{N}\} \subset H_q^1(M) \cap C^\infty(M)$  be an orthonormal basis of the Hilbert space  $L_q^2(M)$  of eigenfunctions of the Dirichlet form (2.7) and let  $\lambda : \mathbb{N} \rightarrow \mathbb{R}^+ \cup \{0\}$  be the corresponding sequence of eigenvalues:

$$(2.10) \quad \lambda_n := \mathcal{Q}(e_n, e_n), \quad \text{for each } n \in \mathbb{N} .$$

The *Friedrichs weighted Sobolev norm*  $\|\cdot\|_s$  of order  $s \in \mathbb{R}^+$  is the norm induced by the hermitian product defined as follows: for all  $u, v \in L_q^2(M)$ ,

$$(2.11) \quad (u, v)_s := \sum_{n \in \mathbb{N}} (1 + \lambda_n)^s \langle u, e_n \rangle_q \langle e_n, v \rangle_q.$$

The inner products (2.4) and (2.11) induce *equivalent* Sobolev norms on the weighted Sobolev space  $H_q^k(M)$ , for all  $k \in \mathbb{Z}^+$ . In fact, the following result, a sharp version of Lemma 4.2 of [For97], holds:

**Lemma 2.5.** *For each  $k \in \mathbb{Z}^+$  there exists a constant  $C_k > 1$  such that, for any orientable holomorphic quadratic differential  $q$  on  $M$  and for all functions  $u \in H_q^k(M)$ ,*

$$(2.12) \quad C_k^{-1} |u|_k \leq \|u\|_k \leq C_k |u|_k.$$

*Proof.* By Proposition 2.4, (3), and Lemma 2.1, for all  $u \in H_q^{k+1}(M)$ ,  $k \in \mathbb{N}$ , the following identity holds (see (4.4) in [For97]):

$$(2.13) \quad \begin{aligned} |u|_{k+1}^2 &= |u|_0^2 + \sum_{i+j \leq k} \mathcal{Q}(S^i T^j u, S^i T^j u) = \\ &= |u|_0^2 + \sum_{i+j \leq k} \langle S^i T^j \partial_q^\pm u, S^i T^j \partial_q^\pm u \rangle_q = |u|_0^2 + |\partial_q^\pm u|_k^2. \end{aligned}$$

Hence in particular  $u \in H_q^{k+1}(M)$ ,  $k \in \mathbb{N}$ , implies  $\partial_q^\pm u \in H_q^k(M)$ . If  $k \geq 1$ , a second application of the identity (2.13) yields

$$(2.14) \quad |u|_{k+1}^2 = |u|_1^2 + |\partial_q^\mp \partial_q^\pm u|_{k-1}^2.$$

The statement then follows by induction on  $k \in \mathbb{N}$ . For  $k = 0$  it is immediate and for  $k = 1$ , by the identity (2.13),

$$(2.15) \quad |u|_1^2 = |u|_0^2 + \mathcal{Q}(u, u) = \sum_{n \in \mathbb{N}} (1 + \lambda_n) |\langle u, e_n \rangle_q|^2.$$

For  $k > 1$ , by induction hypothesis we can assume that the norms  $|\cdot|_{k-1}$  and  $\|\cdot\|_{k-1}$  are equivalent, that is, there exists a constant  $C_{k-1} > 1$  such that, for all  $u \in H_q^{k-1}(M)$ ,

$$(2.16) \quad C_{k-1}^{-1} |u|_{k-1} \leq \|u\|_{k-1} \leq C_{k-1} |u|_{k-1}.$$

Since by (2.3) for all  $u, v \in H_q^1(M)$ ,  $\langle \partial_q^\pm u, v \rangle_q = -\langle u, \partial_q^\mp v \rangle_q$ , the adjoint operator  $(\partial_q^\pm)^* = -\partial_q^\mp$  on  $H_q^1(M)$ . By Proposition 2.4, (3), we have

$$(2.17) \quad \begin{aligned} \|\partial_q^\mp \partial_q^\pm u\|_{k-1}^2 &= \sum_{n \in \mathbb{N}} (1 + \lambda_n)^{k-1} |\langle \partial_q^\pm u, \partial_q^\pm e_n \rangle_q|^2 = \\ &= \sum_{n \in \mathbb{N}} (1 + \lambda_n)^{k-1} |\mathcal{Q}(u, e_n)|^2 = \sum_{n \in \mathbb{N}} (1 + \lambda_n)^{k-1} \lambda_n^2 |\langle u, e_n \rangle_q|^2. \end{aligned}$$

There exists a constant  $C_{k+1} > 1$  such that, for all  $\lambda \geq 0$ ,

$$(2.18) \quad \begin{aligned} C_{k+1}^{-2} (1 + \lambda)^{k+1} &\leq 1 + \lambda + C_{k-1}^{-2} \lambda^2 (1 + \lambda)^{k-1} \\ &\leq 1 + \lambda + C_{k-1}^2 \lambda^2 (1 + \lambda)^{k-1} \leq C_{k+1}^2 (1 + \lambda)^{k+1} . \end{aligned}$$

By (2.14), (2.15), (2.16), (2.17) and (2.18), the estimate

$$(2.19) \quad C_{k+1}^{-1} |u|_{k+1} \leq \|u\|_{k+1} \leq C_{k+1} |u|_{k+1}$$

follows, thereby completing the induction step.  $\square$

**2.2. Fractional Sobolev norms.** Let  $q$  be any orientable quadratic differential on  $M$ . For all  $s \geq 0$ , let

$$(2.20) \quad \bar{H}_q^s(M) := \{u \in L_q^2(M) / \sum_{n \in \mathbb{N}} (1 + \lambda_n)^s |\langle u, e_n \rangle_q|^2 < +\infty\},$$

endowed with the hermitian product given by (2.11) and, for any  $s > 0$ , let  $\bar{H}_q^{-s}(M)$  be the dual space of the Hilbert space  $\bar{H}_q^s(M)$ . The spaces  $\bar{H}_q^s(M)$  will be called the *Friedrichs (fractional) weighted Sobolev spaces*.

Let  $H_1 \subset H_2$  be Hilbert spaces such that  $H_1$  embeds continuously into  $H_2$  with dense image. For all  $\theta \in [0, 1]$ , let  $[H_1, H_2]_\theta$  be the (holomorphic) interpolation space of  $H_1 \subset H_2$  in the sense of Lions-Magenes [LM68], Chap. 1, endowed with the canonical interpolation norm. By the results of [LM68], Chap. 1, §§2, 5, 6 and 14, we have the following:

**Lemma 2.6.** *The Friedrichs weighted Sobolev spaces form an interpolation family  $\{\bar{H}_q^s(M)\}_{s \in \mathbb{R}}$  of Hilbert spaces: for all  $r, s \in \mathbb{R}$  with  $r < s$ ,*

$$(2.21) \quad \bar{H}_q^{(1-\theta)r+\theta s}(M) \equiv [\bar{H}_q^r(M), \bar{H}_q^s(M)]_\theta .$$

The family  $\{H_q^s(M)\}_{s \in \mathbb{R}}$  of *fractional weighted Sobolev spaces* will be defined as follows. Let  $[s] \in \mathbb{N}$  denote the *integer part* and  $\{s\} \in [0, 1)$  the *fractional part* of any real number  $s \geq 0$ .

**Definition 2.7.** (1) The *fractional weighted Sobolev norm*  $|\cdot|_s$  of order  $s \geq 0$  is the euclidean norm induced by the hermitian product defined as follows: for all functions  $u, v \in H_q^\infty(M)$ ,

$$(2.22) \quad \langle u, v \rangle_s := \frac{1}{2} \sum_{i+j \leq [s]} (S^i T^j u, S^i T^j v)_{\{s\}} + (T^i S^j u, T^i S^j v)_{\{s\}} .$$

- (2) The *fractional weighted Sobolev norm*  $|\cdot|_{-s}$  of order  $-s < 0$  is defined as the dual norm of the weighted Sobolev norm  $|\cdot|_s$ .
- (3) The *fractional weighted Sobolev space*  $H_q^s(M)$  of order  $s \in \mathbb{R}$  is defined as the completion with respect to the norm  $|\cdot|_s$  of the maximal common invariant domain  $H_q^\infty(M)$ .

It can be proved that the weighted Sobolev space  $H_q^{-s}(M)$  is isomorphic to the dual space of the Hilbert space  $H_q^s(M)$ , for all  $s \in \mathbb{R}$ .

**Definition 2.8.** For any distribution  $\mathcal{D}$  on  $M \setminus \Sigma_q$ , the *weighted Sobolev order*  $\mathcal{O}_q^H(\mathcal{D})$  and the *Friedrichs weighted Sobolev order*  $\bar{\mathcal{O}}_q^H(\mathcal{D})$  are the real numbers defined as follows:

$$(2.23) \quad \begin{aligned} \mathcal{O}_q^H(\mathcal{D}) &:= \inf\{s \in \mathbb{R} \mid \mathcal{D} \in H_q^{-s}(M)\}; \\ \bar{\mathcal{O}}_q^H(\mathcal{D}) &:= \inf\{s \in \mathbb{R} \mid \mathcal{D} \in \bar{H}_q^{-s}(M)\}. \end{aligned}$$

The definition of the fractional weighted Sobolev norms is motivated by the following basic result.

**Lemma 2.9.** *For all  $s \geq 0$ , the restrictions of the Cauchy-Riemann operators  $\partial_q^\pm : H_q^1(M) \rightarrow L_q^2(M)$  to the subspaces  $H_q^{s+1}(M) \subset H_q^1(M)$  yield bounded operators*

$$\partial_s^\pm : H_q^{s+1}(M) \rightarrow H_q^s(M)$$

*(which do not extend to operators  $\bar{H}_q^{s+1}(M) \rightarrow \bar{H}_q^s(M)$  unless  $M$  is the torus). On the other hand, the Laplace operator*

$$(2.24) \quad \Delta_q = \partial_q^+ \partial_q^- = \partial_q^- \partial_q^+ : H_q^2(M) \rightarrow L_q^2(M)$$

*yields a bounded operator  $\bar{\Delta}_s : \bar{H}_q^{s+2}(M) \rightarrow \bar{H}_q^s(M)$ , defined as the restriction of the Friedrichs extension  $\Delta_q^F : \bar{H}_q^2(M) \rightarrow L_q^2(M)$ .*

*Proof.* The restrictions  $\partial_s^\pm : H_q^{s+1}(M) \rightarrow H_q^s(M)$  of the Cauchy-Riemann operators are well-defined and bounded for all  $s > 0$  by definition of the Sobolev spaces  $H_q^s(M)$ .

The operators  $\partial_s^\pm : H_q^{s+1}(M) \rightarrow H_q^s(M)$  do not extend to bounded operators  $\bar{H}_q^{s+1}(M) \rightarrow \bar{H}_q^s(M)$  unless  $M_q$  is a flat torus. In fact, every finite combination  $f$  of eigenfunctions of the Dirichlet form belongs to  $\bar{H}_q^s(M)$ , for all  $s \in \mathbb{R}$ , but  $\partial_q^\pm f \notin H_q^1(M)$  in all cases because of the presence of obstructions in the Taylor expansion of eigenfunctions at the singular set  $\Sigma_q \neq \emptyset$ . In fact, if  $\partial_q^+ e_n \in H_q^1(M)$  (or  $\partial_q^- e_n \in H_q^1(M)$ ) for all  $n \in \mathbb{N}$ , then  $\partial_q^+ e_n \in H_q^1(M)$  and  $\partial_q^- e_n \in H_q^1(M)$ , for all  $n \in \mathbb{N}$ , since the eigenfunctions  $e_n$  can be chosen real. It follows that  $e_n \in H_q^2(M)$ , for all  $n \in \mathbb{N}$ , hence  $H_q^2(M) = \bar{H}_q^2(M)$  is the domain of the Friedrichs extension  $\Delta_q^F$  of the Laplacian  $\Delta_q$  of the metric  $R_q$ . Thus the Laplacian  $\Delta_q$  is self-adjoint on the domain  $H_q^2(M)$ , hence the metric  $R_q$  has no singularities and  $M$  is the torus. In fact, the space  $H_q^2(M)$  is the domain of the closure of the Laplacian  $\Delta_q$  on the common invariant domain  $H_q^\infty(M)$ . If  $H_q^2(M) = \bar{H}_q^2(M)$ , then  $\Delta_q$  is essentially self-adjoint on  $H_q^\infty(M)$  and by [Nel59], Th. 5 or Cor. 9.1, the action of the commutative Lie algebra spanned by  $\{S, T\}$  integrates to a Lie group action. Hence, the singularity set  $\Sigma_q = \emptyset$  and  $M$  is the torus.

Finally, the Friedrichs extension  $\Delta_q^F$ , defined on  $\bar{H}_q^2(M)$ , has a bounded restriction  $\bar{\Delta}_s : \bar{H}_q^{s+2}(M) \rightarrow \bar{H}_q^s(M)$ , for all  $s \geq 0$ . In fact, we have

$$(2.25) \quad \Delta_q^F u = \sum_{n \in \mathbb{N}} \lambda_n \langle u, e_n \rangle_q e_n, \quad \text{for all } u \in \bar{H}_q^2(M),$$

hence  $\Delta_q^F u \in \bar{H}_q^s(M)$  if  $u \in \bar{H}_q^{s+2}(M)$ , by definition of the Friedrichs weighted Sobolev spaces  $\bar{H}_q^s(M)$  (in terms of eigenfunction expansions for the Dirichlet form).  $\square$

**Lemma 2.10.** *The fractional weighted Sobolev norms satisfy the following interpolation inequalities. For any  $0 \leq r < s$  there exists a constant  $C_{r,s} > 0$  such that, for any  $\theta \in [0, 1]$  and any function  $u \in H_q^s(M)$ ,*

$$(2.26) \quad |u|_{(1-\theta)r+\theta s} \leq C_{r,s} |u|_r^{1-\theta} |u|_s^\theta.$$

*Proof.* The argument will be carried out in three steps: (1) the open interval  $(r, s)$  does not contain integers; (2) the open interval  $(r, s)$  contains a single integer; (3) the general case.

In case (1) there exists  $k \in \mathbb{N}$  such that  $k \leq r < s \leq k+1$ . The interpolation inequality follows from the definition (2.22) of the euclidean product which induces the fractional Sobolev norms, from the interpolation inequality for Friedrichs weighted Sobolev norms (which are by definition interpolation norms) and from the Hölder inequality. In fact, since  $0 \leq r-k \leq s-k \leq 1$  and  $\theta \in (0, 1)$ , the fractional part

$$(2.27) \quad \{(1-\theta)r + \theta s\} = (1-\theta)(r-k) + \theta(s-k),$$

hence, by the interpolation inequality for Friedrichs norms (see for instance [LM68], Chap. 1, §2.5), for all  $i+j \leq k$  the following estimates hold:

$$(2.28) \quad \begin{aligned} |S^i T^j u|_{\{(1-\theta)r+\theta s\}} &\leq |S^i T^j u|_{r-k}^{1-\theta} |S^i T^j u|_{s-k}^\theta; \\ |T^i S^j u|_{\{(1-\theta)r+\theta s\}} &\leq |T^i S^j u|_{r-k}^{1-\theta} |T^i S^j u|_{s-k}^\theta. \end{aligned}$$

By the definition (2.22) of the Sobolev norms, the interpolation inequality (2.26) follows from (2.28) by Hölder inequality.

In case (2), there exists  $k \in \mathbb{N}$  such that  $k-1 \leq r < k < s \leq k+1$ . We claim that, for any  $u \in H_q^s(M)$ ,

$$(2.29) \quad |u|_k \leq C_{r,s} |u|_r^{\frac{s-k}{s-r}} |u|_s^{\frac{k-r}{s-r}}.$$

Let us prove that step (1) and the above claim (2.29) imply step (2). Let  $\sigma = (1-\theta)r + \theta s$ . We will consider only the case when  $\sigma \in (r, k)$  since the case when  $\sigma \in (k, s)$  is similar. By step (1) we have the inequality

$$(2.30) \quad |u|_\sigma \leq C_r |u|_r^{\frac{k-\sigma}{k-r}} |u|_k^{\frac{\sigma-r}{k-r}}.$$



By the claim (2.29) it then follows that

$$(2.31) \quad |u|_\sigma \leq C_{r,s}^{(1)} |u|_r^{\frac{k-\sigma}{k-r} + \frac{\sigma-r}{k-r} \frac{s-k}{s-r}} |u|_s^{\frac{\sigma-r}{k-r} \frac{k-r}{s-r}}.$$

It is immediate to verify that

$$\frac{k-\sigma}{k-r} + \frac{\sigma-r}{k-r} \frac{s-k}{s-r} = \frac{s-\sigma}{s-r}.$$

Let us turn to the proof of the claim (2.29). Since  $-1 \leq r-k < 0 < s-k \leq 1$  and the weighted Sobolev norm  $|\cdot|_s$  coincides with the Friedrichs norm  $\|\cdot\|_s$  for any  $s \in [-1, 1]$ , by the interpolation inequality for the Friedrichs Sobolev norms, the following estimates hold: for all  $i, j \in \mathbb{N}$  with  $i+j \leq k$  and for any function  $u \in H_q^s(M)$ ,

$$(2.32) \quad \begin{aligned} |S^i T^j u|_0 &\leq C_{r,s}^{(2)} |S^i T^j u|_{r-k}^{\frac{s-k}{s-r}} |S^i T^j u|_{s-k}^{\frac{k-r}{s-r}}; \\ |T^i S^j u|_0 &\leq C_{r,s}^{(2)} |T^i S^j u|_{r-k}^{\frac{s-k}{s-r}} |T^i S^j u|_{s-k}^{\frac{k-r}{s-r}}. \end{aligned}$$

Since the operators  $S, T : \bar{H}_q^1(M) \rightarrow L_q^2(M)$  are well-defined, bounded and have bounded linear extensions  $L_q^2(M) \rightarrow \bar{H}_q^{-1}(M)$ , by the fundamental theorem of interpolation (see for instance [LM68], Chap. 1, §5.1), the operators  $S, T : \bar{H}_q^s(M) \rightarrow \bar{H}_q^{s-1}(M)$  are well-defined and bounded for any  $s \in [0, 1]$ . It follows that there exists a constant  $C_r > 0$  such that

$$(2.33) \quad \begin{aligned} \sum_{i+j \leq k} |S^i T^j u|_{r-k}^2 &\leq (C'_r)^2 \sum_{i+j \leq k-1} |S^i T^j u|_{r-(k-1)}^2; \\ \sum_{i+j \leq k} |T^i S^j u|_{r-k}^2 &\leq (C'_r)^2 \sum_{i+j \leq k-1} |T^i S^j u|_{r-(k-1)}^2 \end{aligned}$$

The claim (2.29) then follows by Hölder inequality from (2.32) and (2.33).

In general, let  $k_1 < k_2$  be positive integers such that

$$k_1 - 1 \leq r < k_1 < k_2 < s \leq k_2 + 1.$$

By Lemma 2.5, since the Friedrichs norms are interpolation norms, we have that there exists a constant  $C_{k_1, k_2} > 0$  such that, for all  $k \in \mathbb{N} \cap [k_1, k_2]$ ,

$$(2.34) \quad |u|_k \leq C_{k_1, k_2} |u|_{k_1}^{\frac{k_2-k}{k_2-k_1}} |u|_{k_2}^{\frac{k-k_1}{k_2-k_1}}.$$

By step (2) there exists a constant  $C_{r,s}^{(3)} > 0$  such that

$$(2.35) \quad \begin{aligned} |u|_{k_1} &\leq C_{r,s}^{(3)} |u|_r^{\frac{1}{k_1+1-r}} |u|_{k_1+1}^{\frac{k_1-r}{k_1+1-r}}; \\ |u|_{k_2} &\leq C_{r,s}^{(3)} |u|_{k_2-1}^{\frac{s-k_2}{s-k_2+1}} |u|_s^{\frac{1}{s-k_2+1}}. \end{aligned}$$

The estimates in (2.35) imply, by bootstrap-type estimates based on (2.34) for  $k = k_1 + 1$  and  $k = k_2 - 1$ , that there exists a constant  $C_{r,s}^{(4)} > 0$  such that

$$(2.36) \quad \begin{aligned} |u|_{k_1} &\leq C_{r,s}^{(4)} |u|_r^{\frac{k_2-k_1}{k_2-r}} |u|_{k_2}^{\frac{k_1-r}{k_2-r}}; \\ |u|_{k_2} &\leq C_{r,s}^{(4)} |u|_{k_1}^{\frac{s-k_2}{s-k_1}} |u|_s^{\frac{k_2-k_1}{s-k_1}}. \end{aligned}$$

By (2.36) and again by bootstrap, there exists a constant  $C_{r,s}^{(5)} > 0$  such that

$$(2.37) \quad \begin{aligned} |u|_{k_1} &\leq C_{r,s}^{(5)} |u|_r^{\frac{s-k_1}{s-r}} |u|_s^{\frac{k_1-r}{s-r}}; \\ |u|_{k_2} &\leq C_{r,s}^{(5)} |u|_r^{\frac{s-k_2}{s-r}} |u|_s^{\frac{k_2-r}{s-r}}, \end{aligned}$$

Let  $\sigma \in (r, s)$ . We have proved the interpolation inequality for the subcases  $\sigma = k_1$  and  $\sigma = k_2$ . Let us prove that the general case can be reduced to these subcases. If  $\sigma \in (r, k_1)$ , by step (1) there exists  $C_r''' > 0$  such that

$$(2.38) \quad |u|_\sigma \leq C_r''' |u|_r^{\frac{k_1-\sigma}{k_1-r}} |u|_{k_1}^{\frac{\sigma-r}{k_1-r}}.$$

The interpolation inequality in this case follows immediately from (2.37) and (2.38). If  $\sigma \in (k_2, s)$ , the argument is similar. If  $\sigma \in (k_1, k_2)$ , then by step (1), there exists  $C_{[\sigma]} > 0$  such that

$$(2.39) \quad |u|_\sigma \leq C_{[\sigma]} |u|_{[\sigma]}^{1-\{\sigma\}} |u|_{[\sigma]+1}^{\{\sigma\}}.$$

The interpolation inequality then follows from (2.34), (2.37) and (2.39).  $\square$

Let  $H^s(M)$ ,  $s \in \mathbb{R}$ , denote a family of standard Sobolev spaces on the compact manifold  $M$  (defined with respect to a Riemannian metric). The comparison lemma below clarifies to some extent the relations between the different scales of fractional Sobolev spaces.

**Lemma 2.11.** *The following continuous embedding and isomorphisms of Banach spaces hold:*

- (1)  $H^s(M) \subset H_q^s(M) \equiv \bar{H}_q^s(M)$ , for  $0 \leq s < 1$ ;
- (2)  $H^s(M) \equiv H_q^s(M) \equiv \bar{H}_q^s(M)$ , for  $s = 1$ ;
- (3)  $H_q^s(M) \subset \bar{H}_q^s(M) \subset H^s(M)$ , for  $s > 1$ .

For  $s \in [0, 1]$ , the space  $H^s(M)$  is dense in  $H_q^s(M)$  and, for  $s > 1$ , the closure of  $H_q^s(M)$  in  $\bar{H}_q^s(M)$  or  $H^s(M)$  has finite codimension.

*Proof.* By definition  $H^0(M) = L^2(M)$  and  $H_q^0(M) = \bar{H}_q^0(M) = L_q^2(M)$ . Since the area form induced by any quadratic differential is smooth on  $M$ , which is a compact surface, it follows that  $L^2(M) \subset L_q^2(M)$ . The embedding  $H_q^1(M) \subset \bar{H}_q^1(M)$  follows by Lemma 2.5 and the embedding

$\bar{H}_q^1(M) \subset H_q^1(M)$  holds since the eigenfunctions of the Dirichlet form are in  $H_q^1(M)$ . The isomorphism  $H^1(M) \equiv H_q^1(M)$  is proved in [For02], §6.2. Hence (2) is proved and (1) follows by interpolation.

Let  $s > 1$ . If  $[s] = 2k$  is even, there exists a constant  $A_k > 0$  such that, for all functions  $u \in H_q^\infty(M)$ , we have

$$(2.40) \quad \|u\|_s^2 = \|(I - \Delta_q^F)^k u\|_{\{s\}}^2 \leq A_k^2 \sum_{i+j \leq 2k} \|S^i T^j u\|_{\{s\}}^2 = C_k^2 |u|_s.$$

If  $[s] = 2k+1$  is odd, we argue as follows. The Cauchy-Riemann operators  $\partial_q^\pm : H_q^1(M) \rightarrow L_q^2(M)$  are bounded and extend by duality to bounded operators  $\partial_0^\pm : L_q^2(M) \rightarrow H_q^{-1}(M)$ . Hence, by the fundamental theorem of interpolation (see [LM68], Chap. 1, §5.1), for all  $\sigma \in [0, 1]$  the Cauchy-Riemann operators have bounded restrictions

$$(2.41) \quad \partial_\sigma^\pm : \bar{H}_q^\sigma(M) \rightarrow \bar{H}_q^{\sigma-1}(M).$$

It follows that there exists a constant  $B_k > 0$  such that, for all functions  $u \in H_q^\infty(M)$ , we have

$$(2.42) \quad \|u\|_s^2 = \|(I - \Delta_q^F)^{k+1} u\|_{\{s\}-1}^2 \leq B_k^2 \sum_{i+j \leq 2k+1} \|S^i T^j u\|_{\{s\}}^2 = C_k^2 |u|_s.$$

Thus the embeddings  $H_q^s(M) \subset \bar{H}_q^s(M)$  hold for all  $s > 1$ .

It was proved in [For02], §6.2, that  $H_q^k(M) \subset H^k(M)$ , for all  $k \in \mathbb{Z}^+$ . We prove below the stronger statement that  $\bar{H}_q^s(M) \subset H^s(M)$ , for all  $s \in \mathbb{R}^+$ . Let  $R$  be a smooth Riemannian metric on  $M$  conformally equivalent to the degenerate metric  $R_q$  and let  $H_R^s(M)$ ,  $s \geq 0$ , denote the Sobolev spaces of the Riemannian manifold  $(M, R)$  which are defined as the domains of the powers of the essentially self-adjoint Laplacian  $\Delta_R$  of the metric. Since  $M$  is compact, the Sobolev spaces  $H_R^s(M) \equiv H^s(M)$  are independent, as topological vector spaces, of the choice of the Riemannian metric  $R$ , for all  $s \in \mathbb{R}$ . We claim that  $\bar{H}_q^{2k}(M) \subset H_R^{2k}(M)$ , for all  $k \in \mathbb{Z}^+$ . In fact, there exists a smooth non-negative real-valued function  $W$  on  $M$  (vanishing only at  $\Sigma_q$ ) such that  $W\Delta_q^F \subset \Delta_R$ . Let  $W$  be the unique function such that the area forms of the metrics are related by the identity  $\omega_q = W\omega_R$ . If  $u \in \bar{H}_q^{2k}(M)$ , then  $\Delta_q^F u \in L_q^2(M)$ , so that

$$(2.43) \quad \Delta_R u = W\Delta_q^F u \in L^2(M, \omega_R).$$

Let us assume that  $\bar{H}_q^{2k-2}(M) \subset H_R^{2k-2}(M)$  and let  $u \in \bar{H}_q^{2k}(M)$ . We have

$$(2.44) \quad \Delta_R^k u = \Delta_R^{k-1} W\Delta_q^F u = [\Delta_R^{k-1}, W]\Delta_q^F u + W\Delta_R^{k-1}\Delta_q^F u.$$

Since the commutator  $[\Delta_R^{k-1}, W]$  and  $\Delta_R^{k-1}$  are differential operators of order  $2k-2$  on  $M$  and  $\Delta_q^F u \in H_R^{2k-2}(M)$  by the induction hypothesis, the

function  $\Delta_R^k u \in L^2(M, \omega_R)$ . The claim is therefore proved. It follows by interpolation that  $\bar{H}_q^s(M) \subset H_R^s(M)$ , for all  $s \geq 1$ . Thus (3) is proved.

For  $s \in [0, 1]$ , the space  $C_0^\infty(M \setminus \Sigma_q) \subset H_q^s(M)$  is dense in  $H_q^s(M)$ . For  $s > 1$ , the subset  $C^\infty(M) \cap \bar{H}_q^s(M)$  is dense in  $\bar{H}_q^s(M)$ , since the eigenfunctions of the Dirichlet form (hence all finite linear combinations) belong to  $C^\infty(M)$  and the space  $C^\infty(M)$  is dense in  $H^s(M)$ . Finally, the subspace  $C^\infty(M) \cap H_q^k(M) \subset C^\infty(M)$  can be described, for any  $k \in \mathbb{N}$ , as the kernel of a finite number of distributions of finite order supported on the finite set  $\Sigma_q$  (see [For02], (7.9)), hence for any  $k > s$  the closure of  $H_q^k(M) \subset H_q^s(M)$  in  $\bar{H}_q^s(M)$  or in  $H^s(M)$  has finite codimension.  $\square$

**2.3. Local analysis.** For each  $p \in M$  and all  $k \in \mathbb{Z}^+$ , let  $H_q^k(p)$ ,  $\bar{H}_q^k(p)$ , and  $H^k(p)$  the spaces of germs of functions at  $p$  which belong to  $H_q^k(M)$ ,  $\bar{H}_q^k(M)$  and  $H^k(M)$ , endowed with the respective direct limit topologies. More precisely, a germ of function  $f$  at  $p$  belongs to the space  $H_q^k(p)$ ,  $\bar{H}_q^k(p)$  or  $H^k(p)$  iff it can be realized by a function  $F$  on  $M$  which belongs to the space  $H_q^k(M)$ ,  $\bar{H}_q^k(M)$  or  $H^k(M)$  respectively and the open sets in  $H_q^k(p)$ ,  $\bar{H}_q^k(p)$  or  $H^k(p)$  are defined as the images of open sets in  $H_q^k(M)$ ,  $\bar{H}_q^k(M)$  or  $H^k(M)$  under the natural maps  $H_q^k(M) \rightarrow H_q^k(p)$ ,  $\bar{H}_q^k(M) \rightarrow \bar{H}_q^k(p)$  or  $H^k(M) \rightarrow H^k(p)$ .

By Lemma 2.11 we have the inclusions

$$(2.45) \quad \begin{aligned} H^0(p) &\subset H_q^0(p) \subset \bar{H}_q^0(p); \\ H_q^1(p) &= \bar{H}_q^1(p) = H^1(p); \\ H_q^k(p) &\subset \bar{H}_q^k(p) \subset H^k(p). \end{aligned}$$

If  $p \notin \Sigma_q$ , since there is an open neighbourhood  $D_p$  of  $p$  in  $M$  isomorphic to a flat disk and the operator  $\Delta_q^F$  is elliptic of order 2 on  $D_p$  (isomorphic to the flat Laplacian), all the inclusions in (2.45) are identities. We will describe precisely the inclusions  $H_q^k(p) \subset \bar{H}_q^k(p) \subset H^k(p)$  for  $k > 1$ .

Let  $p \in \Sigma_q$  be a zero of (even) order  $2m$  of the (orientable) quadratic differential  $q$  on  $M$ . There exists a unique canonical holomorphic coordinate  $z : D_p \rightarrow \mathbb{C}$ , defined on a neighbourhood  $D_p$  of  $p \in M$ , such that  $z(p) = 0$  and  $q(z) = z^{2m} dz^2$ . With respect to the canonical coordinate the Cauchy-Riemann operators  $\partial_q^\pm$  can be written in the following form:

$$(2.46) \quad \partial_q^+ = \frac{2}{z^m} \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad \partial_q^- = \frac{2}{z^m} \frac{\partial}{\partial z}.$$

Let  $C^\infty(p)$  be the space of germs at  $p \in M$  of smooth complex-valued functions on  $M$  and for any  $u \in C^\infty(p)$  let

$$(2.47) \quad u(z, \bar{z}) = \sum_{i,j \in \mathbb{N}} a_{ij}(u, p) z^i \bar{z}^j .$$

be its (formal) Taylor series at  $p$  (with respect to the canonical coordinate).

**Lemma 2.12.** *Let  $p \in \Sigma_q$  be a zero of (even) order  $2m$  of the (orientable) quadratic differential  $q$  on  $M$ . For any  $k \in \mathbb{N}$ , a germ*

*$u \in C^\infty(p) \cap \bar{H}_q^k(p) \Leftrightarrow a_{ij}(u, p) = 0$ , for all  $i + j \leq (k - 1)(m + 1)$ , except all pairs  $(i, j)$  for which one of the following conditions holds:*

$$(2.48) \quad \begin{aligned} (1) & \ i \in \mathbb{N} \cdot (m + 1), \ j \in \mathbb{N} \cdot (m + 1); \\ (2) & \ i \in \mathbb{N} \cdot (m + 1), \ j \notin \mathbb{N} \cdot (m + 1) \text{ and } i < j; \\ (3) & \ i \notin \mathbb{N} \cdot (m + 1), \ j \in \mathbb{N} \cdot (m + 1) \text{ and } i > j. \end{aligned}$$

*Proof.* Let  $u \in C^\infty(p)$ . For any  $n \in \mathbb{N}$ , there is a local Taylor expansion

$$u(z, \bar{z}) = \sum_{i+j \leq n} a_{ij}(u, p) z^i \bar{z}^j + R_n^u(z, \bar{z})$$

where the remainder  $R_n^u$  is a smooth function vanishing at order  $n$  at  $p$ . A straightforward calculation (based on formulas (2.46)) yields that any smooth function  $R$  vanishing at  $p$  at order  $n$  belongs to the space  $H_q^k(p) \subset \bar{H}_q^k(p)$  if  $n > (k - 1)m$ . It follows that  $u \in \bar{H}_q^k(p)$  iff its Taylor polynomial of any order  $n > (k - 1)m$  does. The argument can therefore be reduced to the case of polynomials.

It follows from formulas (2.46) that, for all  $\ell \in \mathbb{N}$  and all  $(i, j) \in \mathbb{N} \times \mathbb{N}$ , there exists a complex constant  $c_{ij}^{m, \ell}$  such that

$$(2.49) \quad \Delta_q^\ell(z^i \bar{z}^j) = c_{ij}^{m, \ell} z^{i-\ell(m+1)} \bar{z}^{j-\ell(m+1)} .$$

The area form of the quadratic differential  $q$  can be written as

$$(2.50) \quad \omega_q = |z|^{2m} dx \wedge dy$$

with respect to the canonical coordinate  $z := x + iy$ . Hence, straightforward computations in polar coordinates yield that, if  $c_{ij}^{m, \ell} \neq 0$ ,

$$(2.51) \quad \begin{aligned} \Delta_q^\ell(z^i \bar{z}^j) &\in H_q^0(p) \Leftrightarrow i + j - 2\ell(m + 1) > -(m + 1); \\ \Delta_q^\ell(z^i \bar{z}^j) &\in H_q^1(p) \Leftrightarrow i + j - 2\ell(m + 1) > 0. \end{aligned}$$

If  $c_{ij}^{m, \ell} = 0$ , then either  $i \in \mathbb{N} \cdot (m + 1)$  and  $i < \ell(m + 1)$  or  $j \in \mathbb{N} \cdot (m + 1)$  and  $j < \ell(m + 1)$ . It follows that, if  $i, j \notin \mathbb{N} \cdot (m + 1)$ , then

$$(2.52) \quad z^i \bar{z}^j \in \bar{H}_q^k(p) \Leftrightarrow i + j - (k - 1)(m + 1) > 0 .$$

If  $i \in \mathbb{N} \cdot (m+1)$ ,  $i = h(m+1)$ , and  $j \notin \mathbb{N} \cdot (m+1)$ , then conditions (2.51) apply for all  $\ell \leq h$ , hence (2.52) holds if  $k \leq 2h$ , while if  $k > 2h$ ,

$$(2.53) \quad z^i \bar{z}^j \in \bar{H}_q^k(p) \Leftrightarrow \Delta_q^h(z^i \bar{z}^j) \in H_q^1(p) \Leftrightarrow j > i.$$

Similarly, if  $j \in \mathbb{N} \cdot (m+1)$ ,  $j = h(m+1)$ , and  $i \notin \mathbb{N} \cdot (m+1)$ , then (2.52) holds if  $k \leq 2h$ , while if  $k > 2h$ ,

$$(2.54) \quad z^i \bar{z}^j \in \bar{H}_q^k(p) \Leftrightarrow \Delta_q^h(z^i \bar{z}^j) \in H_q^1(p) \Leftrightarrow i > j.$$

It follows immediately from (2.52), (2.53) and (2.54) that the conditions listed in the statement of the lemma are sufficient. The necessity follows from the following argument. For any  $r_1 < r_2$ , let  $D(r_1, r_2) \subset M$  be the annulus (centered at  $p$ ) defined by the inequalities  $r_1 < |z| < r_2$ . The system of Laurent monomials  $\{z^i \bar{z}^j | i, j \in \mathbb{Z}\}$  is orthogonal in  $\bar{H}^k(D_{r_1, r_2})$ . In fact, a computation in polar coordinates shows that, for all  $(i, j) \neq (i', j') \in \mathbb{Z} \times \mathbb{Z}$ ,

$$(2.55) \quad \int_{D(r_1, r_2)} \Delta_q^\ell(z^i \bar{z}^j) \Delta_q^{\ell'}(\bar{z}^{i'} z^{j'}) \omega_q = 0,$$

hence

$$(2.56) \quad \left\| \sum_{i+j \leq n} a_{ij} z^i \bar{z}^j \right\|_k^2 = \sum_{i+j \leq n} |a_{ij}|^2 \|z^i \bar{z}^j\|_k^2.$$

It follows that only Laurent monomials  $z^i \bar{z}^j \in \bar{H}^k(p)$  can appear in the Taylor expansion of a function  $f \in C^\infty(p) \cap \bar{H}^k(p)$ .  $\square$

**Lemma 2.13.** *Let  $p \in \Sigma_q$  be a zero of order  $2m$  of the quadratic differential  $q$  on  $M$ . For any  $k \in \mathbb{N}$ , a germ*

$$u \in C^\infty(p) \cap H_q^k(p) \Leftrightarrow a_{ij}(u, p) = 0, \quad \text{for all } i+j \leq (k-1)(m+1),$$

*except all pairs  $(i, j) \in \mathbb{N} \cdot (m+1) \times \mathbb{N} \cdot (m+1)$ .*

*Proof.* The proof is similar to that of Lemma 2.12 above. In fact, formulas (2.49) are replaced by the following formulas. For all  $\alpha \in \mathbb{N}$  and all  $i \in \mathbb{N}$ , there exists a complex constant  $c_i^{m, \alpha}$  such that

$$(2.57) \quad (\partial_q^+)^{\alpha} (\partial_q^-)^{\beta} z^i \bar{z}^j = c_i^{m, \alpha} \bar{c}_j^{m, \beta} z^{i-\alpha(m+1)} \bar{z}^{j-\beta(m+1)}.$$

As in the proof of Lemma 2.12, it follows by a straightforward computation in polar coordinates that, if  $c_i^{m, \alpha} \bar{c}_j^{m, \beta} \neq 0$ ,

$$(2.58) \quad (\partial_q^+)^{\alpha} (\partial_q^-)^{\beta} z^i \bar{z}^j \in H_q^0(p) \Leftrightarrow i+j - (\alpha+\beta)(m+1) > -(m+1).$$

Since there exists  $\alpha \in \mathbb{N}$  such that  $c_i^{m, \alpha} = 0$  iff  $i \in \mathbb{N} \cdot (m+1)$ , either  $i, j \in \mathbb{N} \cdot (m+1)$ , in which case  $z^i \bar{z}^j \in H_q^k(p)$ , or there exists  $(\alpha, \beta)$  such that  $\alpha + \beta = k$  and  $c_i^{m, \alpha} \bar{c}_j^{m, \beta} \neq 0$ , in which case  $z^i \bar{z}^j \in H_q^k(p)$  iff  $i+j > (k-1)(m+1)$ .  $\square$

Let  $p \in \Sigma_q$  be a zero of order  $2m_p$  of the orientable quadratic differential  $q$ . Let  $\mathcal{T}_p \subset \mathbb{N} \times \mathbb{N}$  be the set of  $(i, j)$  such that

$$(2.59) \quad \begin{aligned} & i \in \mathbb{N} \cdot (m_p + 1), \quad j \notin \mathbb{N} \cdot (m_p + 1) \quad \text{and} \quad i < j \quad \text{or} \\ & i \notin \mathbb{N} \cdot (m_p + 1), \quad j \in \mathbb{N} \cdot (m_p + 1) \quad \text{and} \quad i > j, \end{aligned}$$

For any  $(i, j) \in \mathcal{T}_p$ , let  $\delta_p^{ij}$  be the linear functional (distribution) on  $C^\infty(M)$  defined as follows. Let  $u(z, \bar{z}) = \sum a_{ij}(u, p) z^i \bar{z}^j$  denote the Taylor expansion of  $u \in C^\infty(M)$  at  $p \in \Sigma_q$  with respect to the canonical coordinate  $z : D_p \rightarrow \mathbb{C}$  for the differential  $q$  at  $p$ . Let

$$(2.60) \quad \delta_p^{ij}(u) := a_{ij}(u, p).$$

It is clear from the definition that  $\delta_p^{ij} = \overline{\delta_p^{ji}}$  for all  $(i, j) \in \mathcal{T}_p$ . A calculation shows that, for any  $h \in \mathbb{N} \setminus \mathbb{N} \cdot (m_p + 1)$  we have the following representation in terms of the Cauchy principal value: for any  $u \in C^\infty(p)$ ,

$$(2.61) \quad \begin{aligned} \delta_p^{h0}(u) &= -\frac{1}{4\pi h} \text{PV} \int_M \frac{\Delta_q u}{z^h} \omega_q; \\ \delta_p^{0h}(u) &= -\frac{1}{4\pi h} \text{PV} \int_M \frac{\Delta_q u}{\bar{z}^h} \omega_q. \end{aligned}$$

(The above formulas can be derived one from the other by conjugation). In addition, for any  $\ell \in \mathbb{N}$ , by formulas (2.49) there exist complex constants  $c_{0,h}^{m_p,\ell} \neq 0$  and  $c_{h,0}^{m_p,\ell} \neq 0$  such that the following identities hold in the sense of distributions:

$$(2.62) \quad \begin{aligned} c_{0,h}^{m_p,\ell} \delta_p^{\ell(m_p+1), \ell(m_p+1)+h} &= \Delta_q^\ell (\delta_p^{0,h}); \\ c_{h,0}^{m_p,\ell} \delta_p^{\ell(m_p+1)+h, \ell(m_p+1)} &= \Delta_q^\ell (\delta_p^{h,0}). \end{aligned}$$

Let  $\mathcal{T}_p^k \subset \mathcal{T}_p$  be the subset of  $(i, j)$  such that  $i + j \leq (k - 1)(m_p + 1)$ .

**Lemma 2.14.** *For each  $(i, j) \in \mathcal{T}_p^k$ , the functional  $\delta_p^{ij}$  has a unique (non-trivial) continuous extension to the space  $\bar{H}_q^k(p)$  and the following holds:*

$$(2.63) \quad H_q^k(p) = \{u \in \bar{H}_q^k(p) \mid \delta_p^{ij}(u) = 0 \text{ for all } (i, j) \in \mathcal{T}_p^k\}.$$

*Proof.* The functions  $z^{-h}$  and  $\bar{z}^{-h} \in L_q^2(D)$  for all  $1 \leq h \leq m_p$  and the operator  $\Delta_q^F : \bar{H}_q^2(p) \rightarrow L_q^2(p)$  is bounded. Hence the linear functionals  $\delta_p^{0h}$  and  $\delta_p^{h0}$  are continuous on  $\bar{H}_q^2(p)$  for all  $1 \leq h \leq m_p$ . Similarly, the distributions  $\text{PV}(z^{-h})$  and  $\text{PV}(\bar{z}^{-h}) \in H^{-1}(D_p)$  for all  $m_p < h < 2(m_p + 1)$  and the operator  $\Delta_q^F : \bar{H}_q^3(p) \rightarrow H^1(p)$  is bounded. Hence the linear functionals  $\delta_p^{0h}$  and  $\delta_p^{h0}$  are continuous on  $\bar{H}_q^3(p)$  for all  $m_p < h < 2(m_p + 1)$ . Since the space  $H_q^k(p)$  is equal to the closure in  $\bar{H}_q^k(p)$  of the subspace  $C^\infty(p) \cap H_q^k(p)$ , the statement for  $k = 2, k = 3$  follows from Lemmas 2.12 and 2.13.

We complete the argument by induction on  $k \in \mathbb{N}$ . The Friedrichs extension defines bounded operators  $\Delta_q^F : \bar{H}_q^{k+1}(p) \rightarrow \bar{H}_q^{k-1}(p)$  and its dual  $\Delta_q^F : \bar{H}_q^{-k+1}(p) \rightarrow \bar{H}_q^{-k-1}(p)$ . By the induction hypothesis, since all functionals in  $\mathcal{T}_p^{k-1}$  extend (uniquely) to bounded functionals in  $\bar{H}_q^{-k+1}(p)$ , it follows that all functionals in  $\Delta_q^F(\mathcal{T}_p^{k-1})$  extend (uniquely) to bounded functionals in  $\bar{H}_q^{-k-1}(p)$  and the following holds. For any  $u \in \bar{H}_q^{k+1}(p)$ ,

$$(2.64) \quad \Delta_q^F u \in H_q^{k-1}(p) \Leftrightarrow \delta_p^{ij}(u) = 0 \quad \text{for all } \delta_p^{ij} \in \Delta_q^F(\mathcal{T}_p^{k-1}).$$

Let  $E_q^{k+1} \subset \bar{H}_q^{k+1}(p)$  the closed finite-codimensional subspace defined as

$$(2.65) \quad E_q^{k+1} := \{u \in \bar{H}_q^{k+1}(p) \mid \delta_p^{ij}(u) = 0 \quad \text{for all } \delta_p^{ij} \in \Delta_q^F(\mathcal{T}_p^{k-1})\}.$$

By formulas (2.62), any distribution  $\delta_p \in \mathcal{T}_p^{k+1} \setminus \Delta_q^F(\mathcal{T}_p^{k-1})$  is of the form  $\delta_p = \delta_p^{0h}$  or  $\delta_p = \delta_p^{h0}$  with  $1 \leq h \leq k(m_p + 1)$ . By formulas (2.61), such distributions have a (unique) continuous extension to the subspace  $E_q^{k+1} \subset \bar{H}_q^{k+1}(p)$ . In fact, the distributions  $\text{PV}(z^{-h})$  and  $\text{PV}(\bar{z}^{-h}) \in H_q^{-k+1}(D_p)$ , for all  $1 \leq h < k(m_p + 1)$ , and  $\Delta_q^F(f) \in H_q^{k-1}(D)$  for all  $f \in E_q^{k+1}$ . Hence all distributions in the set  $\mathcal{T}_p^{k+1}$  have a continuous extension to the space  $\bar{H}_q^{k+1}(p)$  and the characterization (2.63) of the subspace  $H_q^{k+1}(p) \subset \bar{H}_q^{k+1}(p)$  follows by Lemmas 2.12 and 2.13.  $\square$

Let  $\mathcal{D}_q^k$  be the set of all continuous extensions to the space  $\bar{H}_q^k(M)$  of the functionals  $\delta_p^{ij}$  for all  $p \in \Sigma_q$  and all  $(i, j) \in \mathcal{T}_p^k$ . Let

$$(2.66) \quad \mathcal{D}_q := \bigcup_{k \in \mathbb{N}} \mathcal{D}_q^k.$$

**Theorem 2.15.** *The (closed) kernel of the system  $\mathcal{D}_q^k$  on  $\bar{H}_q^k(M)$  coincides with the subspace  $H_q^k(M)$ , that is,*

$$H_q^k(M) = \{u \in \bar{H}_q^k(M) \mid \delta(u) = 0, \quad \text{for all } \delta \in \mathcal{D}_q^k\}.$$

**2.4. Smoothing operators.** We will establish below finer results on the Sobolev regularity of the distributions in  $\mathcal{D}_q$ . The key step will be to construct smoothing operators for the scale of Sobolev spaces  $\{H_q^k(p) \mid k \in \mathbb{N}\}$ .

For any  $p \in \Sigma_q$ , let  $z : D_p \rightarrow \mathbb{C}$  be a canonical coordinate for the (orientable) quadratic differential  $q$  such that  $p \in D_p$  and  $z(p) = 0$ . For any  $(i, j) \in \mathbb{N} \times \mathbb{N}$ , let  $Z_p^{ij} \in C^\infty(M)$  be a function such that

$$(2.67) \quad Z_p^{ij}(z) \equiv z^i \bar{z}^j \quad \text{on } D_p.$$

**Lemma 2.16.** *Let  $p \in \Sigma_q$  be a zero of order  $2m$  of the quadratic differential  $q$  on  $M$ . There exists a one-parameter family  $\{K_p(\tau) \mid \tau \in (0, 1]\}$  of bounded operators  $K_p(\tau) : C^\infty(M) \rightarrow H_q^\infty(M)$  such that the following*



estimates hold. For each  $k \in \mathbb{N}$ , there exists a constant  $C_k > 0$  such that, for any  $(i, j) \in \mathbb{N} \times \mathbb{N}$  and for all  $\tau \in (0, 1]$ :

$$(2.68) \quad \begin{aligned} |K_p(\tau)(Z_p^{ij}) - Z_p^{ij}|_k &\leq C_k \tau^{1+\frac{i+j}{m+1}-k}, & \text{for } k < 1 + \frac{i+j}{m+1}; \\ |K_p(\tau)(Z_p^{ij})|_k &\leq C_k |\log \tau|^{1/2}, & \text{for } k = 1 + \frac{i+j}{m+1}; \\ |K_p(\tau)(Z_p^{ij})|_k &\leq C_k \tau^{-[k-(1+\frac{i+j}{m+1})]}, & \text{for } k > 1 + \frac{i+j}{m+1}. \end{aligned}$$

*Proof.* Let  $z : D_p \rightarrow \mathbb{C}$  be a canonical coordinate at  $p \in \Sigma_q$  defined on an open neighbourhood  $D_p \subset M$  such that  $D_p \cap \Sigma_q = \{p\}$ . There exists  $r_1 > 0$  such that  $D(r_1) \subset\subset z(D_p)$ , where  $D(r_1)$  is the euclidean disk centered at the origin of radius  $r_1 > 0$ . Let  $0 < r_2 < r_1$  and let  $\phi : \mathbb{C} \rightarrow \mathbb{R}$  be any non-negative smooth function identically zero on the closure of  $D(r_2)$  and identically equal to 1 outside  $D(r_1)$ . Let  $D'_p \subset\subset D_p$  be any relatively compact neighbourhood of  $p$  in  $D_p$  such that  $D(r_1) \subset\subset z(D'_p)$ . For any  $\tau \in (0, 1]$ , let us define, for all  $F \in C^\infty(M)$ ,

$$(2.69) \quad K_p(\tau)(F)(x) := \begin{cases} \phi(\tau^{-\frac{1}{m+1}}z(x)) F(z(x)), & \text{for } x \in D_p; \\ F(x), & \text{for } x \notin D'_p. \end{cases}$$

Clearly, the definition (2.69) is well-posed for all  $\tau \in (0, 1]$  and the functions  $K_p(\tau)(F) \in H_q^\infty(M)$ , since the rescaled functions  $\phi_\tau : \mathbb{C} \rightarrow \mathbb{R}$  defined as

$$\phi_\tau(z) := \phi(\tau^{-\frac{1}{m+1}}z), \quad \text{for } z \in \mathbb{C},$$

are smooth and have support away from the origin. Since the functions  $\phi_\tau$  are bounded uniformly with respect to the parameter  $\tau \in (0, 1]$ , for any function  $F \in C^\infty(M) \subset L_q^2(M)$ , by the dominated convergence theorem,

$$(2.70) \quad |K_p(\tau)(F) - F|_0 \rightarrow 0, \quad \text{as } \tau \rightarrow 0^+.$$

Let us denote for convenience, for any  $(a, b) \in \mathbb{N} \times \mathbb{N}$ ,

$$\phi_\tau^{ab}(z) := [(\partial_q^+)^a (\partial_q^-)^b \phi](\tau^{-\frac{1}{m+1}}z), \quad \text{for } z \in \mathbb{C}.$$

The functions  $\phi_\tau^{ab}$  are smooth and bounded on  $\mathbb{C}$ , uniformly with respect to  $\tau > 0$ . For any  $(a, b)$  such that  $(a, b) \neq (0, 0)$ , the function  $\phi_\tau^{ab}$  has compact support contained in the euclidean annulus centered at the origin of inner radius  $r_1 \tau^{\frac{1}{m+1}}$  and outer radius  $r_2 \tau^{\frac{1}{m+1}}$ . For  $a = b = 0$ ,  $\phi_\tau^{ab} = \phi_\tau$  has support outside the euclidean disk of radius  $r_1 \tau^{\frac{1}{m+1}}$  and is identically equal to 1 outside the disk of radius  $r_2 \tau^{\frac{1}{m+1}}$ .

For any  $(i, j) \in \mathbb{N} \times \mathbb{N}$ , let  $K_\tau^{ij} := K_p(\tau)(Z_p^{ij}) \in H_q^\infty(M)$ . A calculation shows that the following formulas hold. For all  $m, i$  and  $\alpha \in \mathbb{N}$  and all

$a \in \mathbb{N}$  such that  $0 \leq a \leq \alpha$ , let

$$(2.71) \quad C_{\alpha,a}^{m,i} = \binom{\alpha}{a} \prod_{\ell=1}^{\alpha-a} [i - \ell(m+1)].$$

For any  $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$ , the derivative  $(\partial_q^+)^{\alpha} (\partial_q^-)^{\beta} (K_{\tau}^{ij})(z)$  is given on  $D_p \setminus \{p\}$  by the sum

$$(2.72) \quad \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} C_{\alpha,a}^{m,i} C_{\beta,b}^{m,j} \phi_{\tau}^{ab}(z) \tau^{-(a+b)} z^{i-(\alpha-a)(m+1)} \bar{z}^{j-(\beta-b)(m+1)}.$$

If  $i + j > (\alpha + \beta - 1)(m + 1)$ , since the functions

$$z^{i-(\alpha-a)(m+1)} \bar{z}^{j-(\beta-b)(m+1)} \in L_q^2(D_p),$$

for all  $0 \leq a \leq \alpha$  and  $0 \leq b \leq \beta$ , and  $\phi_{\tau}^{ab}$  is bounded, by change of variables we obtain that for  $(a, b) \neq (0, 0)$  there exists a constant  $C_{a,b} > 0$  such that

$$(2.73) \quad |\phi_{\tau}^{ab}(z) \tau^{-(a+b)} z^{i-(\alpha-a)(m+1)} \bar{z}^{j-(\beta-b)(m+1)}|_0 \leq C_{a,b} \tau^{1+\frac{i+j}{m+1}-(\alpha+\beta)},$$

and that similarly, for  $a = b = 0$ , there exists a constant  $C_0 > 0$

$$(2.74) \quad |(\phi_{\tau}(z) - 1) z^{i-\alpha(m+1)} \bar{z}^{j-\beta(m+1)}|_0 \leq C_0 \tau^{1+\frac{i+j}{m+1}-(\alpha+\beta)}.$$

If  $i + j < [(\alpha - a) + (\beta - b) - 1](m + 1)$ , there exists a constant  $C'_{a,b} > 0$  such that

$$(2.75) \quad |\phi_{\tau}^{ab}(z) \tau^{-(a+b)} z^{i-(\alpha-a)(m+1)} \bar{z}^{j-(\beta-b)(m+1)}|_0 \leq C'_{a,b} \tau^{1+\frac{i+j}{m+1}-(\alpha+\beta)},$$

since the function  $\phi_{\tau}^{ab}$  is bounded and supported outside the euclidean disk of radius  $r_1 \tau^{\frac{1}{m+1}}$  centered at the origin.

If  $i + j = [(\alpha - a) + (\beta - b) - 1](m + 1)$ , a similar calculation yields

$$(2.76) \quad \begin{aligned} & |\phi_{\tau}^{ab}(z) \tau^{-(a+b)} z^{i-(\alpha-a)(m+1)} \bar{z}^{j-(\beta-b)(m+1)}|_0 \\ & \leq C'_{a,b} \tau^{1+\frac{i+j}{m+1}-(\alpha+\beta)} |\log \tau|^{\frac{1}{2}}. \end{aligned}$$

By formula (2.72) for the Cauchy-Riemann iterated derivatives, the required estimates (2.68) follow immediately from estimates (2.73), (2.74), (2.75) and (2.76).  $\square$

We derive below estimates for the local smoothing family  $\{K_p(\tau)\}$  constructed in Lemma 2.16 with respect to the fractional weighted Sobolev norms. Let  $p \in \Sigma_q$  be any zero of order  $2m_p$  of the quadratic differential

$q$  on  $M$ . For each pair  $(i, j) \in \mathbb{N} \times \mathbb{N}$ , let  $e_p^{ij} : \mathbb{R}^+ \rightarrow [0, 1]$  denote the function defined as follows:

$$(2.77) \quad e_p^{ij}(s) := \begin{cases} \{s\}, & \text{if } [s] = 1 + \frac{i+j}{m_p+1}; \\ 1 - \{s\}, & \text{if } [s] = \frac{i+j}{m_p+1}; \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 2.17.** *The family  $\{K_p(\tau) \mid \tau \in (0, 1]\}$  of local smoothing operators  $K_p(\tau) : C^\infty(M) \rightarrow H_q^\infty(M)$ , defined in (2.69), has the following properties. For each  $s \in \mathbb{R}^+$ , there exists a constant  $C_s > 0$  such that, for any pair  $(i, j) \in \mathbb{N} \times \mathbb{N}$  and all  $\tau \in (0, 1]$ :*

$$\begin{aligned} |K_p(\tau)(Z_p^{ij}) - Z_p^{ij}|_s &\leq C_s \tau^{1 + \frac{i+j}{m_p+1} - s} |\log \tau|^{\frac{e_p^{ij}(s)}{2}}, \quad \text{for } s < 1 + \frac{i+j}{m_p+1}; \\ |K_p(\tau)(Z_p^{ij})|_s &\leq C_s |\log \tau| |\log \tau|^{\frac{e_p^{ij}(s)}{2}}, \quad \text{for } s = 1 + \frac{i+j}{m_p+1}; \\ |K_p(\tau)(Z_p^{ij})|_s &\leq C_s \tau^{-[s - (1 + \frac{i+j}{m_p+1})]} |\log \tau|^{\frac{e_p^{ij}(s)}{2}}, \quad \text{for } s > 1 + \frac{i+j}{m_p+1}. \end{aligned}$$

*Proof.* For any  $k \in \mathbb{N}$ , the function  $Z_p^{ij} \in H_q^k(M)$ , if  $k < 1 + (i+j)/(m+1)$ , since  $z^i \bar{z}^j \in H_q^k(p)$ . By Lemma 2.16, there exists a constant  $C_k > 0$  such that, for all  $\tau \in (0, 1]$ ,

$$(2.78) \quad \begin{aligned} |K_p^{ij}(\tau) - K_p^{ij}(\tau/2)|_k &\leq |(K_p^{ij}(\tau) - Z_p^{ij}) - (K_p^{ij}(\tau/2) - Z_p^{ij})|_k; \\ &\leq |K_p^{ij}(\tau) - Z_p^{ij}|_k + |K_p^{ij}(\tau/2) - Z_p^{ij}|_k \leq C_k \tau^{1 + \frac{i+j}{m+1} - k}. \end{aligned}$$

If  $k \geq 1 + (i+j)/(m+1)$ ,

$$(2.79) \quad |K_p^{ij}(\tau) - K_p^{ij}(\tau/2)|_k \leq |K_p^{ij}(\tau)|_k + |K_p^{ij}(\tau/2)|_k$$

hence, by Lemma 2.16,

$$(2.80) \quad \begin{aligned} |K_p^{ij}(\tau) - K_p^{ij}(\tau/2)|_k &\leq 2C_k |\log \tau|^{1/2}, \quad \text{if } k = 1 + \frac{i+j}{m+1}; \\ |K_p^{ij}(\tau) - K_p^{ij}(\tau/2)|_k &\leq 2C_k \tau^{1 + \frac{i+j}{m+1} - k}, \quad \text{if } k > 1 + \frac{i+j}{m+1}. \end{aligned}$$

As a consequence, by the interpolation inequality (Lemma 2.10), for every  $s \in \mathbb{R}^+$  there exists a constant  $C_s > 0$  such that, for all  $\tau \in (0, 1]$ ,

$$\begin{aligned} |K_p^{ij}(\tau) - K_p^{ij}(\tau/2)|_s &\leq C_s \tau^{1 + \frac{i+j}{m+1} - s}, \quad \text{if } [s], [s] + 1 \neq 1 + \frac{i+j}{m+1}; \\ |K_p^{ij}(\tau) - K_p^{ij}(\tau/2)|_s &\leq C_s \tau^{1 + \frac{i+j}{m+1} - s} |\log \tau|^{\frac{1 - \{s\}}{2}}, \quad \text{if } [s] = 1 + \frac{i+j}{m+1}; \\ |K_p^{ij}(\tau) - K_p^{ij}(\tau/2)|_s &\leq C_s \tau^{1 + \frac{i+j}{m+1} - s} |\log \tau|^{\frac{\{s\}}{2}}, \quad \text{if } [s] = \frac{i+j}{m+1}. \end{aligned}$$

If  $s < 1 + (i + j)/(m + 1)$  and  $[s] \neq (i + j)/(m + 1)$ , for all  $n \in \mathbb{N}$ ,

$$(2.81) \quad |K_p^{ij}(\tau/2^n) - K_p^{ij}(\tau/2^{n+1})|_s \leq C_s \tau^{1+\frac{i+j}{m+1}-s} 2^{-n(1+\frac{i+j}{m+1}-s)},$$

hence, for any fixed  $\tau \in (0, 1]$ , the sequence  $\{K_p^{ij}(\tau/2^n)\}_{n \in \mathbb{N}}$  is Cauchy, and therefore convergent, in the Hilbert space  $H_q^s(M)$ . By Lemma 2.16  $\{K_p^{ij}(\tau/2^n)\}_{n \in \mathbb{N}}$  converges to  $Z_p^{ij}$  in  $H_q^{[s]}(M)$ . Since  $H_q^{[s]}(M) \subset H_q^s(M)$ , by uniqueness of the limit  $Z_p^{ij} \in H_q^s(M)$  and  $\{K_p^{ij}(\tau/2^n)\}_{n \in \mathbb{N}}$  converges to  $Z_p^{ij}$  in  $H_q^s(M)$ . The estimate (2.81) also implies that

$$(2.82) \quad |K_p^{ij}(\tau) - Z_p^{ij}|_s \leq \sum_{n \in \mathbb{N}} |K_p^{ij}(\tau/2^n) - K_p^{ij}(\tau/2^{n+1})|_s \leq C'_s \tau^{1+\frac{i+j}{m+1}-s}.$$

If  $s < 1 + (i + j)/(m + 1)$  and  $[s] = (i + j)/(m + 1)$ , by a similar argument we again get that  $Z_p^{ij} \in H_q^s(M)$  and

$$(2.83) \quad |K_p^{ij}(\tau) - Z_p^{ij}|_s \leq C'_s \tau^{1+\frac{i+j}{m+1}-s} |\log \tau|^{\frac{\{s\}}{2}}.$$

If  $s \geq 1 + (i + j)/(m + 1)$  and  $[s], [s] + 1 \neq 1 + (i + j)/(m + 1)$  we argue as follows. For each  $\tau \leq 1/2$ , we have

$$(2.84) \quad |K_p^{ij}(2\tau) - K_p^{ij}(\tau)|_s \leq C_s (2\tau)^{1+\frac{i+j}{m+1}-s},$$

hence if  $\tau \leq 2^{-n}$ , for all  $0 \leq k < n$ ,

$$(2.85) \quad |K_p^{ij}(2^{k+1}\tau) - K_p^{ij}(2^k\tau)|_s \leq C_s 2^{(k+1)(1+\frac{i+j}{m+1}-s)} \tau^{1+\frac{i+j}{m+1}-s}.$$

It follows that, there exists a constant  $C'_s > 0$  such that

$$(2.86) \quad |K_p^{ij}(2^n\tau) - K_p^{ij}(\tau)|_s \leq C'_s 2^{n(1+\frac{i+j}{m+1}-s)} \tau^{1+\frac{i+j}{m+1}-s}.$$

For every  $\tau \in (0, 1]$ , let  $n(\tau)$  be the maximum  $n \in \mathbb{N}$  such that  $2^n\tau \leq 1$ . By this definition it follows that  $1/2 < 2^{n(\tau)}\tau \leq 1$ . Since

$$\sup_{1/2 \leq \tau \leq 1} |K_p^{ij}(\tau)|_s \leq \sup_{1/2 \leq \tau \leq 1} |K_p^{ij}(\tau)|_{[s]+1} < +\infty,$$

it follows that, there exists a constant  $C''_s > 0$  such that

$$(2.87) \quad \begin{aligned} |K_p^{ij}(\tau)|_s &\leq C''_s \tau^{1+\frac{i+j}{m+1}-s}, & \text{if } s > 1 + \frac{i+j}{m+1}; \\ |K_p^{ij}(\tau)|_s &\leq C''_s |\log \tau|, & \text{if } s = 1 + \frac{i+j}{m+1}. \end{aligned}$$

By a similar argument, for  $s > 1 + (i + j)/(m + 1)$  we have

$$(2.88) \quad \begin{aligned} |K_p^{ij}(\tau)|_s &\leq C''_s \tau^{1+\frac{i+j}{m+1}-s} |\log \tau|^{\frac{1-\{s\}}{2}}, & \text{if } [s] = 1 + \frac{i+j}{m+1}; \\ |K_p^{ij}(\tau)|_s &\leq C''_s \tau^{1+\frac{i+j}{m+1}-s} |\log \tau|^{\frac{\{s\}}{2}}, & \text{if } [s] = \frac{i+j}{m+1}, \end{aligned}$$

while for  $s = 1 + (i + j)/(m + 1)$  we have

$$(2.89) \quad \begin{aligned} |K_p^{ij}(\tau)|_s &\leq C_s'' |\log \tau| |\log \tau|^{\frac{1-\{s\}}{2}}, \quad \text{if } [s] = 1 + \frac{i+j}{m+1}; \\ |K_p^{ij}(\tau)|_s &\leq C_s'' |\log \tau| |\log \tau|^{\frac{\{s\}}{2}}, \quad \text{if } [s] = \frac{i+j}{m+1}. \end{aligned}$$

□

Theorem 2.17 implies in particular the following smoothness results.

**Corollary 2.18.** *Let  $z : D_p \rightarrow C$  be a canonical coordinate for an orientable quadratic differential  $q$  at a zero  $p \in \Sigma_q$  of order  $2m$ . For each  $(i, j) \in \mathbb{N} \times \mathbb{N}$ , the function*

$$(2.90) \quad z^i \bar{z}^j \in H_q^s(p), \quad \text{for all } s < 1 + \frac{i+j}{m_p+1}.$$

**Corollary 2.19.** *Let  $p \in \Sigma_q$  be a zero of order  $2m_p$  and let  $(i, j) \in \mathcal{T}_p$ . The distribution  $\delta_p^{ij}$  has the following regularity properties:*

$$(2.91) \quad \begin{aligned} \delta_p^{ij} &\in \bar{H}_q^{-s}(p) \quad \text{for } s > 1 + \frac{i+j}{m_p+1}, \\ \delta_p^{ij} &\notin H_q^{-s}(p) \quad \text{for } s < 1 + \frac{i+j}{m_p+1}. \end{aligned}$$

*Proof.* By the formulas (2.61) and by Lemma 2.14, for any  $h \in \mathbb{N} \setminus \mathbb{N} \cdot (m_p + 1)$  and for any  $\ell \in \mathbb{N}$ , there exist constants  $C_{h0}^{m_p, \ell} \neq 0$  and  $C_{0h}^{m_p, \ell} \neq 0$  such that the following identities hold in the dual Hilbert space  $\bar{H}_q^{-k}(p)$  for any integer  $k \geq 1 + h/(m_p + 1)$ :

$$(2.92) \quad \begin{aligned} C_{h0}^{m_p, \ell} \delta_p^{h0} &= \Delta_q^{\ell+1} (z^{\ell(m_p+1)-h} \bar{z}^{\ell(m_p+1)}) , \\ C_{0h}^{m_p, \ell} \delta_p^{0h} &= \Delta_q^{\ell+1} (z^{\ell(m_p+1)} \bar{z}^{\ell(m_p+1)-h}) . \end{aligned}$$

By Corollary 2.18, if  $\ell(m_p + 1) - h > 0$ , for all  $s < 2\ell + 1 - h/(m_p + 1)$ ,

$$z^{\ell(m_p+1)-h} \bar{z}^{\ell(m_p+1)} \quad \text{and} \quad z^{\ell(m_p+1)} \bar{z}^{\ell(m_p+1)-h} \in H_q^s(p) \subset \bar{H}_q^s(p).$$

Hence  $\delta_p^{h0}, \delta_p^{0h} \in \bar{H}_q^{-s}(p)$  for all  $s > 1 + h/(m_p + 1)$ . By formulas (2.62) it then follows that  $\delta_p^{ij} \in \bar{H}_q^{-s}(p)$  for all  $s > 1 + (i + j)/(m_p + 1)$  as claimed.

Let  $(i, j) \in \mathcal{T}_p$  and  $s < 1 + (i + j)/(m_p + 1)$ . By Corollary 2.18, the function  $z^i \bar{z}^j \in H_q^s(p)$ . Since by definition  $H_q^\infty(p)$  is dense in  $H_q^s(p)$  for any  $s > 0$ , the functional  $\delta_p^{ij} \equiv 0$  on  $H_q^\infty(p)$  and  $\delta_p^{ij}(z^i \bar{z}^j) = 1$ , it follows that  $\delta_p^{ij}$  does not extend to a bounded functional on  $H_q^s(p)$ . □

Let  $p \in \Sigma_q$  be a zero of (even) order  $2m_p$  the quadratic differential  $q$  on  $M$ . For every  $s \in \mathbb{R}^+$ , let  $\mathcal{T}_p^s \subset \mathcal{T}_p$  be the subset defined as

$$(2.93) \quad \mathcal{T}_p^s := \{(i, j) \in \mathcal{T}_p \mid i + j < (s - 1)(m_p + 1)\}.$$

Let  $\mathcal{D}_q^s \subset \bar{H}_q^{-s}(M)$  be the set of distributions defined as follows:

$$(2.94) \quad \mathcal{D}_q^s := \{\delta_p^{ij} \mid p \in \Sigma_q \text{ and } (i, j) \in \mathcal{T}_p^s\}.$$

**Corollary 2.20.** *The closure of the subspace  $H_q^s(M)$  in  $\bar{H}_q^s(M)$  is a subset of the (closed) kernel of the system  $\mathcal{D}_q^s$  on  $\bar{H}_q^s(M)$ , that is,*

$$(2.95) \quad \overline{H_q^s(M)} \subset \{u \in \bar{H}_q^s(M) \mid \delta(u) = 0, \text{ for all } \delta \in \mathcal{D}_q^s\}$$

*The reverse inclusion holds if the following sufficient condition is satisfied:*

$$(2.96) \quad s \notin \{1 + (i + j)/(m_p + 1) \mid p \in \Sigma_q \text{ and } (i, j) \in \mathcal{T}_p\}.$$

*Proof.* Since  $H_q^\infty(M)$  is dense in  $H_q^s(M) \subset \bar{H}_q^s(M)$ ,

$$H_q^\infty(M) \subset \{u \in \bar{H}_q^s(M) \mid \delta(u) = 0, \text{ for all } \delta \in \mathcal{D}_q^s\}$$

and  $\mathcal{D}_q^s \subset \bar{H}_q^{-s}(M)$ , it follows that

$$\overline{H_q^s(M)} \subset \{u \in \bar{H}_q^s(M) \mid \delta(u) = 0, \text{ for all } \delta \in \mathcal{D}_q^s\}.$$

Conversely, if condition (2.96) is satisfied, by Corollary 2.18 the subspace

$$\{u \in C^\infty(M) \mid \delta(u) = 0, \text{ for all } \delta \in \mathcal{D}_q^s\} \subset H_q^s(M).$$

Since  $C^\infty(M) \cap \bar{H}_q^s(M)$  is dense in  $\bar{H}_q^s(M)$ , the result follows.  $\square$

The regularity result proved in Corollary 2.18 extends to a certain subset of all pairs  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$  if the functions  $z^i \bar{z}^j$  are interpreted as distributions in the sense of the Cauchy principal value:

$$(2.97) \quad \text{PV} \left( z^i \bar{z}^j \right) (v) := \text{PV} \int_M z^i \bar{z}^j v \omega_q, \quad \text{for all } v \in C^\infty(p).$$

The most general regularity result for the distributions (2.97) is based on the following generalization of Corollary 2.18 to include logarithmic factors.

**Lemma 2.21.** *Let  $z : D_p \rightarrow \mathbb{C}$  be a canonical coordinate for an orientable holomorphic quadratic differential  $q$  at a zero  $p \in \Sigma_q$  of order  $2m$ . For each  $(i, j, h) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ , the function*

$$z^i \bar{z}^j \log^h |z| \in H_q^s(p), \quad \text{for all } s < 1 + \frac{i + j}{m_p + 1}.$$

*Proof.* Simple calculations show that  $\log |z| \in L_q^2(M)$  and that by formulas (2.46) the following identities hold on  $D_p \setminus \{p\}$ :

$$(2.98) \quad \partial^+ \log |z| = \frac{1}{\bar{z}^{m+1}} \quad \text{and} \quad \partial^- \log |z| = \frac{1}{z^{m+1}}.$$

It follows that, for each  $(i, j, h) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$  and each  $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$ , there exists a finite sequence of non-zero constants  $C_1, \dots, C_h$ , which depend on  $(i, j, h, \alpha, \beta, m)$ , such that the following identity holds on  $D_p \setminus \{p\}$ :

$$(2.99) \quad (\partial^+)^{\alpha} (\partial^-)^{\beta} (z^i \bar{z}^j \log^h |z|) = z^{i-\alpha(m+1)} \bar{z}^{j-\beta(m+1)} \sum_{\ell=0}^h C_{\ell} \log^{\ell} |z|.$$

For all  $(i, j, h) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ , let  $L_p^{ijh} \in C^{\infty}(M \setminus \{p\})$  be any function such that  $L_p^{ijh}(z) = z^i \bar{z}^j \log^h |z|$  for all  $z \in D_p$ . By (2.99), the function

$$L_p^{ijh} \in H_q^k(M), \quad \text{if } k \in \mathbb{N} \text{ and } k < 1 + \frac{i+j}{m_p+1}$$

Let  $\{K_p(\tau) \mid \tau \in (0, 1]\}$  be the family of local smoothing operators defined by formulas (2.69). By computations similar to those carried out in the proof of Lemma 2.16, based on formulas (2.99), it is possible to prove that for each  $k \in \mathbb{N}$ , there exists a constant  $C_k > 0$  such that for all  $\tau \in (0, 1]$ :

$$\begin{aligned} |K_p(\tau)(L_p^{ijh}) - L_p^{ijh}|_k &\leq C_k \tau^{1+\frac{i+j}{m+1}-k} |\log \tau|^h, \quad \text{for } k < 1 + \frac{i+j}{m+1}; \\ |K_p(\tau)(L_p^{ijh})|_k &\leq C_k |\log \tau|^{1/2} |\log \tau|^h, \quad \text{for } k = 1 + \frac{i+j}{m+1}; \\ |K_p(\tau)(L_p^{ijh})|_k &\leq C_k \tau^{-[k-(1+\frac{i+j}{m+1})]} |\log \tau|^h, \quad \text{for } k > 1 + \frac{i+j}{m+1}. \end{aligned}$$

Reasoning as in the proof of Theorem 2.17, we can derive similar estimates for fractional Sobolev norms. For each  $(i, j) \in \mathbb{N} \times \mathbb{N}$ , let  $e_p^{ij} : \mathbb{R}^+ \rightarrow [0, 1]$  be the function defined in formula (2.77). By the interpolation Lemma 2.10, for any  $s < 1 + (i+j)/(m+1)$  there exists a constant  $C_s > 0$  such that

$$|K_p(\tau)(L_p^{ijh}) - K_p(\tau/2)(L_p^{ijh})|_s \leq C_s \tau^{1+\frac{i+j}{m+1}-s} |\log \tau|^{h+\frac{e_p^{ij}(s)}{2}}.$$

It follows that the sequence  $\{K_p(\tau/2^n)(L_p^{ijh})\}_{n \in \mathbb{N}}$  is Cauchy and therefore converges in  $H_q^s(M)$ . By uniqueness of the limit

$$L_p^{ijh} \in H_q^s(M), \quad \text{for all } s < 1 + \frac{i+j}{m_p+1}.$$

In addition, the following estimates hold. For each  $s \in \mathbb{R}^+$  there exists a constant  $C'_s > 0$  such that for all  $\tau \in (0, 1]$ :

$$\begin{aligned} |K_p(\tau)(L_p^{ijh})|_s &\leq C'_s |\log \tau| |\log \tau|^{h + \frac{e_p^{ij}(s)}{2}}, & \text{for } s = 1 + \frac{i+j}{m_p+1}; \\ |K_p(\tau)(L_p^{ijh})|_s &\leq C'_s \tau^{-[s - (1 + \frac{i+j}{m_p+1})]} |\log \tau|^{h + \frac{e_p^{ij}(s)}{2}}, & \text{for } s > 1 + \frac{i+j}{m_p+1}. \end{aligned}$$

□

**Theorem 2.22.** *Let  $z : D_p \rightarrow \mathbb{C}$  be a canonical coordinate for an orientable holomorphic quadratic differential  $q$  at a zero  $p \in \Sigma_q$  of order  $2m_p$ . For each  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$  such that (1)  $i - j \notin \mathbb{Z} \cdot (m_p + 1)$  or (2)  $i > -(m_p + 1)$  or (3)  $j > -(m_p + 1)$ , the distribution*

$$(2.100) \quad \text{PV} (z^i \bar{z}^j \log^h |z|) \in H_q^s(p), \quad \text{for all } s < 1 + \frac{i+j}{m_p+1}.$$

*Proof.* For all  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$  such that  $i - j \notin \mathbb{Z} \cdot (m_p + 1)$  and for any  $h \in \mathbb{Z}$  the following formulas hold for all functions  $v \in H_q^\infty(p)$ :

(2.101)

$$\begin{aligned} (a) \quad \text{PV} \int_M \partial^+ (z^i \bar{z}^j \log^h |z|) v \omega_q &= -\text{PV} \int_M z^i \bar{z}^j \log^h |z| \partial^+ v \omega_q; \\ (b) \quad \text{PV} \int_M \partial^- (z^i \bar{z}^j \log^h |z|) v \omega_q &= -\text{PV} \int_M z^i \bar{z}^j \log^h |z| \partial^- v \omega_q. \end{aligned}$$

Formulas (2.101) also hold in case (a) if  $i > -(m_p + 1)$ ,  $j \in \mathbb{Z}$ , and in case (b) if  $j > -(m_p + 1)$ ,  $i \in \mathbb{Z}$ , for all germs  $v \in C^\infty(p)$ .

By taking into account the formulas (2.46) for the Cauchy-Riemann operators with respect to a canonical coordinate, it follows from formulas (2.101) by induction on  $h \in \mathbb{N}$  that if

$$\text{PV}(z^i \bar{z}^j \log^h |z|) \in H_q^s(p), \quad \text{for all } h \in \mathbb{N},$$

then, if  $i - j \notin \mathbb{Z} \cdot (m_p + 1)$  or  $i > -(m_p + 1)$  and  $j \in \mathbb{Z}$ ,

$$\text{PV}(z^i \bar{z}^{j-(m_p+1)} \log^h |z|) \in H_q^{s-1}(p), \quad \text{for all } h \in \mathbb{N},$$

and, if  $i - j \notin \mathbb{Z} \cdot (m_p + 1)$  or  $j > -(m_p + 1)$  and  $i \in \mathbb{Z}$ ,

$$\text{PV}(z^{i-(m_p+1)} \bar{z}^j \log^h |z|) \in H_q^{s-1}(p), \quad \text{for all } h \in \mathbb{N}.$$

Thus, the statement of the theorem can be derived from Corollary 2.18, by an induction argument based on formulas (2.101). □

**Corollary 2.23.** *Let  $z : D_p \rightarrow \mathbb{C}$  be a canonical coordinate for an orientable holomorphic quadratic differential  $q$  at a zero  $p \in \Sigma_q$  of order  $2m_p$ .*



If  $(i, j) \notin \mathbb{Z} \cdot (m_p + 1) \times \mathbb{Z} \cdot (m_p + 1)$ , the distribution

$$(2.102) \quad \text{PV} (z^i \bar{z}^j \log^h |z|) \notin H_q^s(p), \quad \text{for } s > 1 + \frac{i+j}{m_p+1},$$

and, if  $i - j \in \mathbb{Z} \cdot (m_p + 1)$  and both  $i \leq -(m+1)$  and  $j \leq -(m+1)$ ,

$$(2.103) \quad \text{PV} (z^i \bar{z}^j \log^h |z|) \notin H_q^{-\infty}(p).$$

*Proof.* We argue by contradiction. Assume there exists  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$  such that  $(i, j) \notin \mathbb{Z} \cdot (m_p + 1) \times \mathbb{Z} \cdot (m_p + 1)$  and  $\text{PV} (z^i \bar{z}^j) \in H_q^r(p)$  for some  $r > 1 + (i+j)/(m_p+1)$ . By taking Cauchy-Riemann derivatives if necessary, we can assume that  $i \leq 0$  and  $j \leq 0$ . By Theorem 2.22, the distribution

$$\text{PV} (z^{-i-(m+1)} \bar{z}^{-j-(m+1)}) \in H_q^s(p), \quad \text{for all } s < -1 - \frac{i+j}{m_p+1}.$$

It follows that, for any positive smooth function  $\phi \in C_0^\infty(D_p)$  identically equal to 1 on a disk  $D'_p \subset\subset D_p$  centered at  $p \in \Sigma_q$ , the principal value

$$\text{PV} \int_M \phi(z) z^i \bar{z}^j z^{-i-(m+1)} \bar{z}^{-j-(m+1)} \omega_q$$

is finite, which can be proved to be false by a simple computation in (geodesic) polar coordinates. This contradiction proves the first part of the statement.

If  $i - j \in \mathbb{Z} \cdot (m_p + 1)$  and both  $i \leq -(m+1)$  and  $j \leq -(m+1)$ , we argue as follows. It is not restrictive to assume that  $i \geq j$ , hence the function  $\bar{z}^{i-j} \in H_q^\infty(M)$ . However, by a computation in polar coordinates, since  $i \leq -(m_p + 1)$ ,

$$\text{PV} \int_M \phi(z) z^i \bar{z}^j \log |z| \bar{z}^{i-j} \omega_q = +\infty.$$

It follows that  $\text{PV} (z^i \bar{z}^j \log^h |z|) \notin H_q^{-\infty}(p)$ , hence the second part of the statement is also proved.  $\square$

We conclude this section with a fundamental smoothing theorem for the 1-parameter family of weighted Sobolev spaces.

**Theorem 2.24.** *For each  $k \in \mathbb{N}$ , there exists a family  $\{\mathcal{S}^k(\tau) \mid \tau \in (0, 1]\}$  of bounded operators  $\mathcal{S}^k(\tau) : L_q^2(M) \rightarrow H_q^k(M)$  such that the following estimates hold. For any  $s, r \in [0, k]$  and for any  $\epsilon > 0$ , there exists a constant  $C_{r,s}^k(\epsilon) > 0$  such that, for all  $u \in H_q^s(M)$  and for all  $\tau \in (0, 1]$ :*

$$(2.104) \quad \begin{aligned} |\mathcal{S}^k(\tau)(u) - u|_r &\leq C_{r,s}^k(\epsilon) \|u\|_s \tau^{s-r-\epsilon}, & \text{if } s > r; \\ |\mathcal{S}^k(\tau)(u)|_r &\leq C_{r,s}^k(\epsilon) \|u\|_s \tau^{s-r-\epsilon}, & \text{if } s \leq r. \end{aligned}$$

*Proof.* For each  $p \in \Sigma_q$ , let  $z : D_p \rightarrow \mathbb{C}$  be a canonical coordinate defined on a disk  $D_p$  (centered at  $p$ ) such that  $D_p \cap \Sigma_q = \{p\}$ . For each  $(i, j) \in \mathcal{T}_p$ , let  $Z_p^{ij} \in C^\infty(M)$  be a (fixed) smooth extension, as in (2.67), of the locally defined function  $z^i \bar{z}^j \in C^\infty(p)$ . Let  $P^k$  be the linear operator defined as follows:

$$(2.105) \quad P^k(f) := f - \sum_{p \in \Sigma_q} \sum_{(i,j) \in \mathcal{T}_p^k} \delta_p^{ij}(f) Z_p^{ij}, \quad \text{for all } f \in \bar{H}_q^k(M).$$

The operator  $P_k : \bar{H}_q^k(M) \rightarrow H_q^k(M)$  is well-defined and bounded. It is well-defined by Lemma 2.14 and Theorem 2.15. It is bounded since, for all  $p \in \Sigma_q$  and all  $(i, j) \in \mathcal{T}_p$ , the functions  $Z_p^{ij} \in \bar{H}_q^\infty(M)$  by Lemma 2.12 and, for all  $(i, j) \in \mathcal{T}_p^\ell$ , the distributions  $\delta_p^{ij} \in \bar{H}_q^{-k}(M)$  by Corollary 2.19. In fact, the condition  $(i, j) \in \mathcal{T}_p^\ell$  implies  $i + j < (k - 1)(m_p + 1)$ .

For each  $p \in \Sigma_q$ , let  $\{K_p(\tau) \mid \tau \in (0, 1]\}$  be the family of local smoothing operators constructed in Lemma 2.16. Let  $\{S_\tau^k \mid \tau \in (0, 1]\}$  be the one-parameter family of bounded linear operators  $S_\tau^k : \bar{H}_q^k(M) \rightarrow H_q^k(M)$  defined as follows. For all  $f \in \bar{H}_q^k(M)$ , we let

$$(2.106) \quad S_\tau^k(f) := P^k(f) + \sum_{p \in \Sigma_q} \sum_{(i,j) \in \mathcal{T}_p^k} \delta_p^{ij}(f) K_p(\tau) (Z_p^{ij}).$$

By definition the following identity holds for all  $f \in \bar{H}_q^k(M)$ :

$$S_\tau^k(f) - f = \sum_{p \in \Sigma_q} \sum_{(i,j) \in \mathcal{T}_p^k} \delta_p^{ij}(f) [K_p(\tau) (Z_p^{ij}) - Z_p^{ij}].$$

Since for all  $p \in \Sigma_q$  the condition  $(i, j) \in \mathcal{T}_p$  implies  $i + j \notin \mathbb{N} \cdot (m_p + 1)$ , by Lemma 2.16 the following estimate holds. For each  $\ell \in \mathbb{N}$  such that  $\ell \leq k$ , there exists a constant  $C_\ell^k > 0$  such that, for any  $f \in \bar{H}_q^k(M)$ ,

$$(2.107) \quad \|S_\tau^k(f) - f\|_\ell \leq C_\ell^k \sum_{p \in \Sigma_q} \sum_{(i,j) \in \mathcal{T}_p^k} \tau^{1 + \frac{i+j}{m_p+1} - \ell} |\delta_p^{ij}(f)|.$$

In fact, for each  $p \in \Sigma_q$  and each  $(i, j) \in \mathcal{T}_p$ , since  $Z_p^{ij} \in H_q^\ell(M)$ , which implies  $K_p(\tau) (Z_p^{ij}) - Z_p^{ij} \in H_q^\ell(M)$ , if  $\ell < 1 + (i + j)/(m_p + 1)$ , by Lemma 2.16, there exist constants  $C'_\ell > 0$ ,  $C''_\ell > 0$  such that

$$\|K_p(\tau) (Z_p^{ij}) - Z_p^{ij}\|_\ell \leq C'_\ell \|K_p(\tau) (Z_p^{ij}) - Z_p^{ij}\| \leq C''_\ell \tau^{1 + \frac{i+j}{m_p+1} - \ell},$$

while, if  $\ell > 1 + (i + j)/(m_p + 1)$ , since  $Z_p^{ij} \in \bar{H}_q^\infty(M)$ ,

$$\|K_p(\tau) (Z_p^{ij}) - Z_p^{ij}\|_\ell \leq \|K_p(\tau) (Z_p^{ij})\|_\ell + \|Z_p^{ij}\|_\ell \leq C''_\ell \tau^{1 + \frac{i+j}{m_p+1} - \ell}.$$

The scale of Friedrichs Sobolev spaces admits a standard family of smoothing operators  $\{T_\tau \mid \tau > 0\}$  such that the operator  $T_\tau : L_q^2(M) \rightarrow \bar{H}_q^\infty(M)$

is defined, for each  $\tau > 0$ , by the following truncation of Fourier series. Let  $\{e_n \mid n \in \mathbb{N}\}$  be an orthonormal basis of eigenfunctions of the Friedrichs Laplacian  $\Delta_q^F$  and let  $\lambda : \mathbb{N} \rightarrow \mathbb{R}^+ \cup \{0\}$  be the corresponding sequence of eigenvalues. Then

$$T_\tau(u) := \sum_{\tau^2 \lambda_n \leq 1} \langle u, e_n \rangle_q e_n, \quad \text{if } u = \sum_{n \in \mathbb{N}} \langle u, e_n \rangle_q e_n.$$

If  $u \in \bar{H}_q^s(M)$ , then  $T_\tau(u) \rightarrow u$  in  $\bar{H}_q^s(M)$  (as  $\tau \rightarrow 0^+$ ) and the following estimates hold. For all  $r \in \mathbb{R}^+$ , there exists a constant  $C_{r,s}(q) > 0$  such that

$$(2.108) \quad \begin{aligned} \|T_\tau(u) - u\|_r &\leq C_{r,s}(q) \|u\|_s \tau^{s-r}, & \text{if } s \geq r; \\ \|T_\tau(u)\|_r &\leq C_{r,s}(q) \|u\|_s \tau^{-(r-s)}, & \text{if } r \geq s. \end{aligned}$$

If  $p \in \Sigma_q$  and  $(i, j) \in \mathcal{T}_p$ , the distribution  $\delta_p^{ij} \in \bar{H}_q^{-s_{ij}}(M)$  for any  $s_{ij} > 0$  such that  $i+j < (s_{ij}-1)(m_p+1)$ . Hence there exists a constant  $C_p^{ij}(q) > 0$  such that, by estimates (2.108), for all  $u \in \bar{H}_q^s(M)$ ,

$$(2.109) \quad |\delta_p^{ij}(T_\tau(u))| \leq C_p^{ij}(q) \|u\|_s \max\{1, \tau^{s-s_{ij}}\}.$$

If  $u \in H_q^s(M)$ , then  $\delta_{ij}^p(u) = 0$ , for all  $p \in \Sigma_q$  and all  $(i, j) \in \mathcal{T}_p^s$ . Hence, if  $s_{ij} \leq s$  and  $i+j < (s_{ij}-1)(m_p+1)$ ,

$$(2.110) \quad |\delta_p^{ij}(T_\tau(u))| = |\delta_p^{ij}(T_\tau(u) - u)| \leq C_p^{ij}(q) \|u\|_s \tau^{s-s_{ij}}.$$

The following estimates hold. Let  $s \in \mathbb{R}^+$  and  $\ell \in \mathbb{N}$ . For any  $\epsilon > 0$ , there exists a constant  $C_{\ell,s}(\epsilon) > 0$  such that, for all  $\tau \in (0, 1]$  and all  $u \in H_q^s(M)$ ,

$$(2.111) \quad \|S_\tau^k \circ T_\tau(u) - T_\tau(u)\|_\ell \leq C_{\ell,s}(\epsilon) \|u\|_s \tau^{s-\ell-\epsilon}.$$

In fact, if  $p \in \Sigma_q$  and  $(i, j) \in \mathcal{T}_p^k$ ,

$$(2.112) \quad |\delta_p^{ij}(T_\tau(u))| \leq C_p^{ij}(q) \|u\|_s \tau^{s-s_{ij}};$$

for any  $s_{ij} > 1 + (i+j)/(m_p+1) \geq s$ , if  $(i, j) \in \mathcal{T}_p^k \setminus \mathcal{T}_p^s$ , and for any  $1 + (i+j)/(m_p+1) < s_{ij} \leq s$ , if  $(i, j) \in \mathcal{T}_p^s$ . The claim (2.111) then follows from (2.107). By estimates (2.108) and (2.111), for any  $\epsilon > 0$ , there exists a constant  $C'_{\ell,s}(\epsilon) > 0$  such that, for all  $\tau \in (0, 1]$  and for all  $u \in H_q^s(M)$ ,

$$(2.113) \quad \|S_\tau^k \circ T_\tau(u) - u\|_\ell \leq C'_{\ell,s}(\epsilon) \|u\|_s \tau^{s-\ell-\epsilon}.$$

Let  $\{\mathcal{S}^k(\tau) \mid \tau \in (0, 1]\}$  be the family of operators  $\mathcal{S}^k(\tau) : L_q^2(M) \rightarrow H_q^k(M)$  defined as follows: for each  $\tau \in (0, 1]$ ,

$$(2.114) \quad \mathcal{S}^k(\tau) := S_\tau^k \circ T_\tau.$$

By estimate (2.113), for any  $\epsilon > 0$ , there exists a constant  $C''_{\ell,s}(\epsilon) > 0$  such that, for all  $\tau \in (0, 1]$  and for all  $u \in H_q^s(M)$ ,

$$(2.115) \quad \|\mathcal{S}^k(\tau)(u) - \mathcal{S}^k(\tau/2)(u)\|_\ell \leq C''_{\ell,s}(\epsilon) \|u\|_s \tau^{s-\ell-\epsilon}.$$

Since  $\mathcal{S}^k(\tau)(u) \in H_q^k(M)$  for all  $\tau \in (0, 1]$ , by the interpolation inequality proved in Lemma 2.10 it follows that, for any  $r \in [0, k]$  and for any  $\epsilon > 0$  there exists  $C_{r,s}(\epsilon) > 0$  such that, for all  $\tau \in (0, 1]$  and for all  $u \in H_q^s(M)$ ,

$$(2.116) \quad |\mathcal{S}^k(\tau)(u) - \mathcal{S}^k(\tau/2)(u)|_r \leq C_{r,s}(\epsilon) \|u\|_s \tau^{s-r-\epsilon}.$$

It follows that, for every  $n \in \mathbb{N}$  and for every  $\tau \in (0, 1]$ ,

$$(2.117) \quad |\mathcal{S}^k(\tau/2^n)(u) - \mathcal{S}^k(\tau/2^{n+1})(u)|_r \leq C_{r,s}(\epsilon) \|u\|_s 2^{-n(s-r-\epsilon)} \tau^{s-r-\epsilon}.$$

If  $s > r$  and  $0 < \epsilon < s - r$ , the sequence  $\{\mathcal{S}^k(\tau/2^n)(u)\}_{n \in \mathbb{N}}$  is Cauchy and therefore convergent to the function  $u \in H_q^s(M) \subset H_q^r(M)$  in the Hilbert space  $H_q^r(M)$ . It follows that, for all  $\tau \in (0, 1]$  and for all  $u \in H_q^s(M)$ ,

$$(2.118) \quad |\mathcal{S}^k(\tau)(u) - u|_r \leq C_{r,s}(\epsilon) \|u\|_s \tau^{s-r-\epsilon}.$$

If  $s \leq r$  and  $s - r \leq 0 < \epsilon$ , we argue as follows. By estimate (2.116), for each  $\tau \leq 1/2$  we have

$$(2.119) \quad |\mathcal{S}^k(2\tau)(u) - \mathcal{S}^k(\tau)(u)|_r \leq C_{r,s}(\epsilon) \|u\|_s (2\tau)^{s-r-\epsilon},$$

hence if  $\tau \leq 2^{-n}$ , for all  $0 \leq k < n$ ,

$$(2.120) \quad |\mathcal{S}^k(2^{k+1}\tau)(u) - \mathcal{S}^k(2^k\tau)(u)|_r \leq C_{r,s}(\epsilon) \|u\|_s 2^{(k+1)(s-r-\epsilon)} \tau^{s-r-\epsilon},$$

It follows that, there exists a constant  $C'_{r,s}(\epsilon) > 0$  such that

$$(2.121) \quad |\mathcal{S}^k(2^n\tau)(u) - \mathcal{S}^k(\tau)(u)|_r \leq C'_{r,s}(\epsilon) \|u\|_s 2^{n(s-r-\epsilon)} \tau^{s-r-\epsilon},$$

For every  $\tau \in (0, 1]$ , let  $n(\tau)$  be the maximum  $n \in \mathbb{N}$  such that  $2^n\tau \leq 1$ . By this definition it follows that  $1/2 < 2^{n(\tau)}\tau \leq 1$ , hence by estimate (2.113) there exists a constant  $C_k > 0$  such that, for all  $u \in H_q^s(M)$ ,

$$|\mathcal{S}^k(2^{n(\tau)}\tau)(u)|_r \leq \sup_{1/2 \leq \tau \leq 1} \|\mathcal{S}^k(\tau)(u)\|_k < C_k \|u\|_s.$$

It follows that there exists  $C''_{r,s}(\epsilon) > 0$  such that, for all  $\tau \in (0, 1]$  and for all  $u \in H_q^s(M)$ ,

$$(2.122) \quad |\mathcal{S}^k(\tau)(u)|_r \leq C''_{r,s}(\epsilon) \|u\|_s \tau^{s-r-\epsilon},$$

□

By Lemma 2.11 and Theorem 2.24, we have the following comparison estimate for the (Friedrichs) weighted Sobolev norms :

**Corollary 2.25.** *For any  $0 < r < s$  there exists constants  $C_r > 0$  and  $C_{r,s} > 0$  such that, for all  $u \in H_q^s(M)$ , the following inequalities hold:*

$$C_r^{-1} \|u\|_r \leq |u|_r \leq C_{r,s} \|u\|_s.$$

Finally, we derive a crucial interpolation estimate for the dual weighted Sobolev norms:

**Corollary 2.26.** *Let  $0 \leq s_1 < s_2$ . For any  $s_1 \leq r < s \leq s_2$  there exists a constant  $C_{r,s} > 0$  such that for any distribution  $u \in H_q^{-s_1}(M)$  the following interpolation inequality holds:*

$$(2.123) \quad |u|_{-s} \leq C_{r,s} |u|_{-s_1}^{\frac{s_2-r}{s_2-s_1}} |u|_{-s_2}^{\frac{r-s_1}{s_2-s_1}}.$$

*Proof.* Let  $k \in \mathbb{N}$  be any integer larger than  $s_2 > s_1$  and let  $\mathcal{S}^k(\tau) : L_q^2(M) \rightarrow H_q^k(M)$  be the family of smoothing operators constructed above. By Theorem 2.24, since  $0 \leq r - s_1 < s - s_1$  and any  $r - s_2 < s - s_2 \leq 0$ , there exists a constant  $C_{r,s}^k > 0$  such that the following holds: for any  $u \in H_q^{-s_1}(M) \setminus \{0\}$ , any  $v \in H_q^s(M)$  and for all  $\tau \in (0, 1]$ ,

$$(2.124) \quad \begin{aligned} |\langle u, v \rangle| &\leq |u|_{-s_1} |v - \mathcal{S}^k(\tau)(v)|_{s_1} + |u|_{-s_2} |\mathcal{S}^k(\tau)(v)|_{s_2} \\ &\leq C_{r,s}^k \{ \tau^{r-s_1} |u|_{-s_1} + \tau^{r-s_2} |u|_{-s_2} \} |v|_s. \end{aligned}$$

The interpolation inequality (2.123) then follows by taking

$$\tau = \left( \frac{|u|_{-s_2}}{|u|_{-s_1}} \right)^{\frac{1}{s_2-s_1}} \in (0, 1].$$

□

### 3. THE COHOMOLOGICAL EQUATION

**3.1. Distributional solutions.** In this section we give a streamlined version of the main argument of [For97] (Theorem 4.1) with the goal of establishing the sharpest bound on the loss of Sobolev regularity within the reach of the methods of [For97]. We were initially motivated by a question of Marmi, Moussa and Yoccoz who found for *almost all* orientable quadratic differentials a loss of regularity of  $1 + \text{BV}$  (they find bounded solutions for absolutely continuous data with first derivative of bounded variation under finitely many independent compatibility conditions and corresponding results for higher smoothness) [MMY03], [MMY05]. The results of this section, as those of [For97], hold for *all* orientable quadratic differentials.

There is a natural action of the circle group  $S^1 \equiv SO(2, \mathbb{R})$  on the space  $Q(M)$  of holomorphic quadratic differentials on a Riemann surface  $M$ :

$$r_\theta(q) := e^{i\theta} q, \quad \text{for all } (r_\theta, q) \in SO(2, \mathbb{R}) \times Q(M).$$

Let  $q_\theta$  denote the quadratic differential  $r_\theta(q)$  and let  $\{S_\theta, T_\theta\}$  denote the frame (introduced in §2.1) associated to the quadratic differential  $q_\theta$  for any

$\theta \in S^1$ . We have the following formulas:

$$(3.1) \quad \begin{aligned} S_\theta &= \cos\left(\frac{\theta}{2}\right) S_q + \sin\left(\frac{\theta}{2}\right) T_q = \frac{e^{-i\frac{\theta}{2}}}{2} \partial_q^+ + \frac{e^{i\frac{\theta}{2}}}{2} \partial_q^-; \\ T_\theta &= -\sin\left(\frac{\theta}{2}\right) S_q + \cos\left(\frac{\theta}{2}\right) T_q = \frac{e^{-i\frac{\theta}{2}}}{2i} \partial_q^+ - \frac{e^{i\frac{\theta}{2}}}{2i} \partial_q^-; \end{aligned}$$

**Definition 3.1.** Let  $q$  be an orientable quadratic differential. A distribution  $u \in \bar{H}_q^{-r}(M)$  will be called a *(distributional) solution* of the cohomological equation  $S_q u = f$  for a given function  $f \in \bar{H}_q^{-s}(M)$  if

$$\langle u, S_q v \rangle = -\langle f, v \rangle, \quad \text{for all } v \in H_q^{r+1}(M) \cap \bar{H}_q^s(M).$$

Let  $\bar{\mathcal{H}}_q^s(M) \subset \bar{H}_q^s(M)$ ,  $\mathcal{H}_q^s(M) \subset H_q^s(M)$  (for any  $s \in \mathbb{R}$ ) be the subspaces orthogonal to constant functions, that is

$$(3.2) \quad \begin{aligned} \bar{\mathcal{H}}_q^s(M) &:= \{f \in \bar{H}_q^s(M) \mid \langle f, 1 \rangle_s = 0\}, \\ \mathcal{H}_q^s(M) &:= \{f \in H_q^s(M) \mid (f, 1)_s = 0\}. \end{aligned}$$

The spaces  $\bar{\mathcal{H}}_q^s(M) \subset \bar{H}_q^s(M)$  and  $\mathcal{H}_q^s(M) \subset H_q^s(M)$  coincide with the subspaces of functions of zero average for  $s \geq 0$ , and with the subspaces of distributions vanishing on constant functions for  $s < 0$ .

**Theorem 3.2.** Let  $r > 2$  and  $p \in (0, 1)$  be such that  $rp > 2$ . There exists a bounded linear operator

$$\mathcal{U} : \bar{\mathcal{H}}_q^{-1}(M) \rightarrow L^p(S^1, \bar{H}_q^{-r}(M))$$

such that the following holds. For any  $f \in \bar{\mathcal{H}}_q^{-1}(M)$  there exists a full measure subset  $\mathcal{F}_r(f) \subset S^1$  such that  $u := \mathcal{U}(f)(\theta) \in \bar{H}_q^{-r}(M)$  is a distributional solution of the cohomological equation  $S_\theta u = f$  for all  $\theta \in \mathcal{F}_r(f)$ .

*Proof.* We claim that for any  $r > 2$ , any  $p \in (0, 1)$  such that  $pr > 2$  and any  $f \in \bar{\mathcal{H}}_q^{-1}(M)$ , there exists a measurable function  $A_q := A_q(p, r, f) \in L^p(S^1, \mathbb{R}^+)$  such that the following estimates hold. Let  $\theta \in S^1$  be such that  $A_q(\theta) < +\infty$ . For all  $v \in H_q^{r+1}(M)$  we have

$$(3.3) \quad |\langle f, v \rangle| \leq A_q(\theta) \|S_\theta v\|_r.$$

In addition, the following bound for the  $L^p$  norm of the function  $A_q$  holds. There exists a constant  $B_q(p) > 0$  such that

$$(3.4) \quad |A_q|_p \leq B_q(p) \|f\|_{-1}.$$

Assuming the claim, we prove the statement of the theorem. In fact, by the estimate (3.3) the linear map given by

$$(3.5) \quad S_\theta v \rightarrow -\langle f, v \rangle, \quad \text{for all } v \in H_q^{r+1}(M),$$

is well defined and extends by continuity to the closure of the range  $\bar{R}_r(\theta)$  of the linear operator  $S_\theta$  in  $\bar{H}_q^r(M)$ . Let  $\mathcal{U}(f)(\theta)$  be the extension uniquely defined by the condition that  $\mathcal{U}(f)(\theta)$  vanishes on the orthogonal complement of  $\bar{R}_r(\theta)$  in  $\bar{H}_q^r(M)$ . By construction, for almost all  $\theta \in S^1$  the linear functional  $u := \mathcal{U}(f)(\theta) \in \bar{H}_q^{-r}(M)$  yields a distributional solution of the cohomological equation  $S_\theta u = f$  whose norm satisfies the bound

$$\|\mathcal{U}(f)(\theta)\|_{-r} \leq A_q(\theta) .$$

By (3.4) the  $L^p$  norm of the measurable function  $\mathcal{U}(f) : S^1 \rightarrow \bar{H}_q^{-r}(M)$  satisfies the required estimate

$$|\mathcal{U}(f)|_p := \left( \int_{S^1} \|\mathcal{U}(f)(\theta)\|_{-r}^p d\theta \right)^{1/p} \leq B_q(p) \|f\|_{-1} .$$

We turn now to the proof of the above claim. Let  $R_q^\pm = (\mathcal{M}_q^\mp)^\perp$  be the (closed) ranges of the Cauchy-Riemann operators  $\partial_q^\pm : H_q^1(M) \rightarrow L_q^2(M)$  (see Proposition 2.4). Following [For97], we introduce the linear operator  $U_q : R_q^- \rightarrow R_q^+$  defined as

$$(3.6) \quad U_q := \partial_q^-(\partial_q^+)^{-1} .$$

By Proposition 2.4, (3), the operator  $U_q$  is a partial isometry on  $L_q^2(M)$ , hence by the standard theory of partial isometries on Hilbert spaces, it has a family of unitary extensions  $U_J : L_q^2(M) \rightarrow L_q^2(M)$  parametrized by isometries  $J : \mathcal{M}_q^+ \rightarrow \mathcal{M}_q^-$  (see formulas (3.10)-(3.12) in [For97]). By definition the following identities hold on  $H_q^1(M)$  (see formulas (3.13) in [For97]) :

$$(3.7) \quad S_\theta = \frac{e^{i\frac{\theta}{2}}}{2} (U_J + e^{-i\theta}) \partial_q^+ = \frac{e^{-i\frac{\theta}{2}}}{2} (U_J^{-1} + e^{i\theta}) \partial_q^- .$$

The proof of estimate (3.3) is going to be based on properties of the resolvent of the operator  $U_J$ . In fact, the proof of (3.3) is based on the results, summarized in [For97], Corollary 3.4, concerning the non-tangential boundary behaviour of the resolvent of a unitary operator on a Hilbert space, applied to the operators  $U_J, U_J^{-1}$  on  $L_q^2(M)$ . The Fourier analysis of [For97], §2, also plays a relevant role through Lemma 4.2 in [For97] and the Weyl's asymptotic formula (Theorem 2.3).

Following [For97], Prop. 4.6A, or [For02], Lemma 7.3, we prove that there exists a constant  $C_q > 0$  such that the following holds. For any distribution  $f \in H_q^{-1}(M)$  there exist (weak) solutions  $F^\pm \in L_q^2(M)$  of the equations  $\partial_q^\pm F^\pm = f$  such that

$$(3.8) \quad |F^\pm|_0 \leq C_q \|f\|_{-1} .$$

In fact, the maps given by

$$(3.9) \quad \partial_q^\pm v \rightarrow -\langle f, v \rangle, \quad \text{for all } v \in H_q^1(M),$$

are bounded linear functionals on the (closed) ranges  $R_q^\pm \subset L_q^2(M)$  (of the Cauchy-Riemann operator  $\partial_q^\pm : H_q^1(M) \rightarrow L_q^2(M)$ ). In fact, the functionals are well-defined since  $f$  vanishes on constant functions, that is, on the kernel of the Cauchy-Riemann operators, and it is bounded since by Poincaré inequality (see [For97], Lemma 2.2 or [For02], Lemma 6.9) there exists a constant  $C_q > 0$  such that, for any  $v \in \mathcal{H}_q^1(M) \subset H_q^1(M)$ ,

$$(3.10) \quad |\langle f, v \rangle| \leq \|f\|_{-1} \|v\|_1 \leq C_q \|f\|_{-1} |\partial_q^\pm v|_0.$$

Let  $\Phi^\pm$  be the unique linear extension of the linear map (3.9) to  $L_q^2(M)$  which vanishes on the orthogonal complement of  $R_q^\pm$  in  $L_q^2(M)$ . By (3.10), the functionals  $\Phi^\pm$  are bounded on  $L_q^2(M)$  with norm

$$\|\Phi^\pm\| \leq C_q \|f\|_{-1}.$$

By the Riesz representation theorem, there exist two (unique) functions  $F^\pm \in L_q^2(M)$  such that

$$\langle v, F^\pm \rangle_q = \Phi^\pm(v), \quad \text{for all } v \in L_q^2(M).$$

The functions  $F^\pm$  are by construction (weak) solutions of the equations  $\partial_q^\pm F^\pm = f$  satisfying the required bound (3.8).

The identities (3.7) immediately imply that

$$(3.11) \quad \begin{aligned} \langle \partial_q^\pm v, F^\pm \rangle_q &= 2e^{\mp i\frac{\theta}{2}} \langle \mathcal{R}_J^\pm(z) S_\theta v, F^\pm \rangle_q \\ &\quad - (z + e^{\mp i\theta}) \langle \mathcal{R}_J^\pm(z) \partial_q^\pm v, F^\pm \rangle_q, \end{aligned}$$

where  $\mathcal{R}_J^+(z)$  and  $\mathcal{R}_J^-(z)$  denote the resolvents of the unitary operators  $U_J$  and  $U_J^{-1}$  respectively, which yield holomorphic families of bounded operators on the unit disk  $D \subset \mathbb{C}$ .

Let  $r > 2$  and let  $p \in (0, 1)$  be such that  $pr > 2$ . Let  $\mathcal{E} = \{e_k\}_{k \in \mathbb{N}}$  be the orthonormal Fourier basis of the Hilbert space  $L_q^2(M)$  described in §2. By Corollary 3.4 in [For97] all holomorphic functions

$$(3.12) \quad \mathcal{R}_k^\pm(z) := \langle \mathcal{R}_J^\pm(z) e_k, F^\pm \rangle_q, \quad k \in \mathbb{N},$$

belong to the Hardy space  $H^p(D)$ , for any  $0 < p < 1$ . The corresponding non-tangential maximal functions  $N_k^\pm$  (over cones of arbitrary fixed aperture  $0 < \alpha < 1$ ) belong to the space  $L^p(S^1, d\theta)$  and for all  $0 < p < 1$  there exists a constant  $A_{\alpha,p} > 0$  such that the following inequalities hold:

$$(3.13) \quad |N_k^\pm|_p \leq A_{\alpha,p} |e_k|_0 |F^\pm|_0 = A_{\alpha,p} |F^\pm|_0 \leq A_{\alpha,p} C_q \|f\|_{-1}.$$



Let  $\{\lambda_k\}_{k \in \mathbb{N}}$  be the sequence of the eigenvalues of the Dirichlet form  $\mathcal{Q}$  introduced in §2. Let  $w \in \bar{H}_q^r(M)$ . We have

$$(3.14) \quad \langle \mathcal{R}_J^\pm(z)w, F^\pm \rangle_q = \sum_{k=0}^{\infty} \langle w, e_k \rangle_q \mathcal{R}_k^\pm(z) ,$$

hence, by the Cauchy-Schwarz inequality,

$$(3.15) \quad |\langle \mathcal{R}_J^\pm(z)w, F^\pm \rangle_q| \leq \left( \sum_{k=0}^{\infty} \frac{|\mathcal{R}_k^\pm(z)|^2}{(1 + \lambda_k)^r} \right)^{1/2} \|w\|_r ,$$

Let  $N^\pm(\theta)$  be the functions defined as

$$(3.16) \quad N^\pm(\theta) := \left( \sum_{k=0}^{\infty} \frac{|N_k^\pm(\theta)|^2}{(1 + \lambda_k)^r} \right)^{1/2} .$$

Let  $N^\pm(w)$  be the non-tangential maximal function for the holomorphic function  $\langle \mathcal{R}_J^\pm(z)w, F^\pm \rangle_q$ . By formulas (3.15) and (3.16), for all  $\theta \in S^1$  and all functions  $w \in \bar{H}_q^r(M)$ , we have

$$(3.17) \quad N^\pm(w)(\theta) \leq N^\pm(\theta) \|w\|_r .$$

The functions  $N^\pm \in L^p(S^1, d\theta)$  for any  $0 < p < 1$ . In fact, by formula (3.13) and (following a suggestion of Stephen Semmes) by the ‘triangular inequality’ for  $L^p$  spaces with  $0 < p < 1$ , we have

$$(3.18) \quad |N^\pm|_p^p \leq (A_{\alpha,p} C_q)^p \left( \sum_{k=0}^{\infty} \frac{1}{(1 + \lambda_k)^{pr/2}} \right) \|f\|_{-1}^p < +\infty .$$

The series in formula (3.18) is convergent by the Weyl asymptotics (Theorem 2.3) since  $pr/2 > 1$ .

By taking the non-tangential limit as  $z \rightarrow -e^{\mp i\theta}$  in the identity (3.11), formula (3.17) implies that, for all  $\theta \in S^1$  such that  $N^\pm(\pi \mp \theta) < +\infty$ ,

$$|\langle \partial^\pm v, F^\pm \rangle_q| \leq N^\pm(\pi \mp \theta) \|S_\theta v\|_r ,$$

hence the required estimates (3.3) and (3.4) are proved with the choice of the function  $A_q(\theta) := N^+(\pi - \theta)$  or  $A_q(\theta) := N^-(\pi + \theta)$  for all  $\theta \in S^1$ . Since the claim is proved the result follows.  $\square$

**Theorem 3.3.** *Let  $r > 2$ . For almost all  $\theta \in S^1$  (with respect to the Lebesgue measure), there exists a constant  $C_r(\theta) > 0$  such that, for all  $f \in \bar{H}_q^{r-1}(M)$  such that  $\int_M f \omega_q = 0$ , the cohomological equation  $S_\theta u = f$  has a distributional solution  $u \in \bar{H}_q^{-r}(M)$  satisfying the following estimate:*

$$\|u\|_{-r} \leq C_r(\theta) \|f\|_{r-1} .$$

*Proof.* Let  $\mathcal{E} = \{e_k\}_{k \in \mathbb{N}}$  be the orthonormal Fourier basis of the Hilbert space  $L_q^2(M)$  described in §2. Let  $r > 2$  and  $p \in (0, 1)$  be such that  $pr > 2$ . By Theorem 3.2, for any  $k \in \mathbb{N} \setminus \{0\}$  there exists a function with distributional values  $u_k := \mathcal{U}(e_k) \in L^p(S^1, \bar{H}_q^{-r}(M))$  such that the following holds. There exists a constant  $C_q := C_q(p, r) > 0$  such that

$$(3.19) \quad \left( \int_{S^1} \|u_k(\theta)\|_{-r}^p d\theta \right)^{1/p} \leq C_q \|e_k\|_{-1} = C_q (1 + \lambda_k)^{-1/2}.$$

In addition, for any  $k \in \mathbb{N} \setminus \{0\}$ , there exists a full measure set  $\mathcal{F}_k \subset S^1$  such that, for all  $\theta \in \mathcal{F}_k$ , the distribution  $u := u_k(\theta) \in \bar{H}_q^{-r}(M)$  is a (distributional) solution of the cohomological equation  $S_\theta u = e_k$ .

Any function  $f \in \bar{H}_q^{r-1}(M)$  such that  $\int_M f \omega_q = 0$  has a Fourier decomposition in  $L_q^2(M)$ :

$$f = \sum_{k \in \mathbb{N} \setminus \{0\}} \langle f, e_k \rangle_q e_k.$$

A (formal) solution of the cohomological equation  $S_\theta u = f$  is therefore given by the series

$$(3.20) \quad u_\theta := \sum_{k \in \mathbb{N} \setminus \{0\}} \langle f, e_k \rangle_q u_k(\theta).$$

By the triangular inequality in  $\bar{H}_q^{-r}(M)$  and by Hölder inequality, we have

$$\|u_\theta\|_{-r} \leq \left( \sum_{k \in \mathbb{N} \setminus \{0\}} \frac{\|u_k(\theta)\|_{-r}^2}{(1 + \lambda_k)^{r-1}} \right)^{1/2} \|f\|_{r-1},$$

hence by the ‘triangular inequality’ for  $L^p$  spaces (with  $0 < p < 1$ ) and by the estimate (3.19),

$$(3.21) \quad \int_{S^1} \|u_\theta\|_{-r}^p d\theta \leq C_q^p \left( \sum_{k \in \mathbb{N} \setminus \{0\}} \frac{1}{(1 + \lambda_k)^{pr/2}} \right) \|f\|_{r-1}^p.$$

Since  $pr/2 > 1$  the series in (3.21) is convergent, hence by Chebyshev inequality for the space  $L^p(S^1)$ , there exists a full measure set  $\mathcal{B} \subset S^1$  such that, for all  $\theta \in \mathcal{B}$ , formula (3.20) yields a well-defined distribution  $u_\theta \in \bar{H}_q^{-r}(M)$  and there exists a constant  $C_q(\theta) > 0$  such that

$$(3.22) \quad \|u_\theta\|_{-r} \leq C_q(\theta) \|f\|_{r-1}.$$

The set  $\mathcal{F} = \cap \mathcal{F}_k \cap \mathcal{B}$  has full measure and for all  $\theta \in \mathcal{F}$ , for all  $k \in \mathbb{N} \setminus \{0\}$ , the distribution  $u_k(\theta) \in \bar{H}_q^{-r}(M)$  is a solution of the equation  $S_\theta u = e_k$ . It follows that  $u_\theta \in \bar{H}_q^{-r}(M)$  is a solution of the cohomological equation  $S_\theta u = f$  which satisfies the required bound (3.22).  $\square$

We finally derive a result on distributional solutions of the cohomological equation for distributional data of arbitrary regularity:

**Corollary 3.4.** *For any  $s \in \mathbb{R}$  there exists  $r > 0$  such that the following holds. For almost all  $\theta \in S^1$  (with respect to the Lebesgue measure), there exists a constant  $C_{r,s}(\theta) > 0$  such that, for all  $F \in \bar{H}_q^s(M)$  orthogonal to constant functions, the cohomological equation  $S_\theta U = F$  has a distributional solution  $U \in \bar{H}_q^{-r}(M)$  satisfying the following estimate:*

$$\|U\|_{-r} \leq C_{r,s}(\theta) \|F\|_s .$$

*Proof.* Since  $F \in \bar{H}_q^s(M)$  is orthogonal to constant functions, for every  $k \in \mathbb{N}$  there exists  $f_k \in \bar{H}_q^{s+2k}(M)$ , orthogonal to constant functions, such that  $(I - \Delta_q^F)^k f_k = F$ . In fact, the family of Friedrichs Sobolev spaces  $\{\bar{H}_q^s(M) | s \in \mathbb{R}\}$  is defined in terms of the Friedrichs extension  $\Delta_q^F$  of the Laplace operator  $\Delta_q$  of the flat metric determined by the quadratic differential. Let  $n \in \mathbb{N}$  be the minimum integer such that  $\sigma := s + 2n + 1 > 2$ . By Theorem 3.3, for almost all  $\theta \in S^1$  (with respect to the Lebesgue measure) the cohomological equation  $S_\theta u = f_n$  has a distributional solution  $u_\theta \in \bar{H}_q^{-\sigma}(M)$  satisfying the following estimate:

$$(3.23) \quad \|u_\theta\|_{-\sigma} \leq C_\sigma(\theta) \|f_n\|_{\sigma-1} .$$

Let  $U_\theta := (I - \Delta_q^F)^n u_\theta \in \bar{H}_q^{-\sigma-2n}(M)$ . It follows immediately from the estimate (3.23) and from the definitions that

$$\|U_\theta\|_{-\sigma-2n} = \|u_\theta\|_{-\sigma} \leq C_\sigma(\theta) \|f_n\|_{\sigma-1} = C_\sigma(\theta) \|F\|_s .$$

Finally  $U_\theta \in \bar{H}_q^{-\sigma-2n}(M)$  is a distributional solution of the cohomological equation  $S_\theta U = F$ , for almost all  $\theta \in S^1$ . In fact, for any  $v \in H_q^{\sigma+2n+1}(M)$ , the function  $S_\theta v \in H_q^{\sigma+2n}(M)$ , hence  $(I - \Delta_q)^n S_\theta v = S_\theta (I - \Delta_q)^n v$  and, since the distribution  $u_\theta \in \bar{H}_q^{-\sigma}(M)$  is a solution of the cohomological equation  $S_\theta u = f_n$ , for almost all  $\theta \in S^1$ , and  $(I - \Delta_q^F)^n v \in H_q^{\sigma+1}(M)$ ,

$$\begin{aligned} \langle U_\theta, S_\theta v \rangle &= \langle (I - \Delta_q^F)^n u_\theta, S_\theta v \rangle = \langle u_\theta, S_\theta (I - \Delta_q)^n v \rangle \\ &= -\langle f_n, (I - \Delta_q^F)^n v \rangle = -\langle (I - \Delta_q^F)^n f_n, v \rangle = -\langle F, v \rangle , \end{aligned}$$

as required by the definition of distributional solution of the cohomological equation (Definition 3.1).  $\square$

**3.2. Invariant distributions and basic currents.** Invariant distributions yield obstructions to the existence of smooth solutions of the cohomological equation. We derive below from Theorem 3.2 a sharp version of the main results of [For02], §6, about the Sobolev regularity of invariant distributions. We then recall the structure theorem proved in that paper on the space of invariant distributions (see [For02], Th. 7.7).

Invariant distributions for the horizontal [respectively vertical] vector field of an orientable quadratic differential  $q$  are closely related to *basic currents* (of dimension and degree equal to 1) for the horizontal [vertical] foliation  $\mathcal{F}_q$  [ $\mathcal{F}_{-q}$ ]. The notion of a basic current for a measured foliation on a Riemann surface has been studied in detail in [For02], §6, in the context of weighted Sobolev spaces with integer exponent. We outline below some of the basic constructions and results on basic currents and invariant distributions which carry over without modifications to the more general context of fractional weighted Sobolev spaces. Finally, we derive from Theorem 3.2 a result on the Sobolev regularity of basic currents (or invariant distributions) which improves upon a similar result proved in [For02] (see Theorem 7.1 (i)).

Let  $\Sigma \subset M$  be a finite subset. The space  $\mathcal{D}(M \setminus \Sigma)$  will denote the standard space of de Rham currents on the open manifold  $M \setminus \Sigma$ , that is the dual of the Fréchet space  $\Omega_c(M \setminus \Sigma)$  of differential forms with compact support in  $M \setminus \Sigma$ . A homogenous current of dimension  $d \in \mathbb{N}$  (and degree  $2 - d$ ) on  $M \setminus \Sigma$  is a continuous linear functional on the subspace  $\Omega_c^d(M \setminus \Sigma)$  of differential forms of degree  $d$ . The subspace of homogeneous currents of dimension  $d$  on  $M \setminus \Sigma$  will be denote by  $\mathcal{D}^d(M \setminus \Sigma)$ .

Let  $q$  be an orientable quadratic differential on a Riemann surface  $M$ . Let  $\Sigma_q$  be the (finite) set of its zeroes. In [For02] we have introduced the following space  $\Omega_q(M)$  of smooth test forms on  $M$ .

**Definition 3.5.** For any  $p \in M$  of (even) order  $k = 2m \in \mathbb{N}$  ( $m = 0$  if  $p \notin \Sigma_q$ ), let  $z : \mathcal{U}_p \rightarrow \mathbb{C}$  be a canonical complex coordinate on a neighbourhood  $\mathcal{U}_p$  of  $p \in M$ , that is a complex coordinate such that  $z(p) = 0$  and  $q = z^k dz^2$  on  $\mathcal{U}_p$ . Let  $\pi_p : \mathcal{U}_p \rightarrow \mathbb{C}$  be the (local) covering map defined by

$$\pi_p(z) := \frac{z^{m+1}}{m+1}, \quad z \in \mathbb{C}.$$

The space  $\Omega_q(M)$  is defined as the space of smooth forms  $\alpha$  on  $M$  such that the following holds: for all  $p \in M$ , there exists a smooth form  $\lambda_p$  on a neighbourhood of  $0 \in \mathbb{C}$  such that  $\alpha = \pi_p^*(\lambda_p)$  on  $\mathcal{U}'_p \subset \mathcal{U}_p$ . The space  $\Omega_q(M)$  is the direct sum of the subspaces  $\Omega_q^d(M)$  of homogeneous forms of degree  $d \in \{0, 1, 2\}$ . The spaces  $\Omega_q^d(M)$ , for any  $d \in \{0, 1, 2\}$ , and  $\Omega_q(M)$  can be endowed with a natural Fréchet topology modeled on the smooth topology in every coordinate neighbourhood.

**Lemma 3.6.** *For any orientable quadratic differential  $q \in Q(M)$ , the space of functions  $\Omega_q^0(M)$  is dense in the space  $H_q^\infty(M)$  endowed with the inverse limit Fréchet topology induced by the family of weighted Sobolev norms.*

*Proof.* By definition, the MacLaurin series of any  $f \in \Omega_q(M)$  with respect to a canonical complex coordinate  $z$  for  $q$  at every  $p \in \Sigma_q$  (of order  $2m_p$ )

has the following form:

$$(3.24) \quad f(z) = \sum_{h,k \in \mathbb{N}} f_{hk} z^{h(m_p+1)} \bar{z}^{k(m_p+1)}.$$

By Lemmas 2.12 and 2.13,  $f \in H_q^\infty(M)$ . Thus  $\Omega_q(M) \subset H_q^\infty(M)$ .

Let  $F \in H_q^\infty(M)$ . By Lemma 2.11 the function  $F \in C^\infty(M)$  and by Lemmas 2.12 and 2.13 its MacLaurin series has the form (3.24) at every  $p \in \Sigma_q$  (of order  $2m_p$ ). By Borel's theorem and by a partition of unity argument, there exists a function  $f \in \Omega_q(M)$  such that  $F - f \in C^\infty(M)$  vanishes at infinite order at  $\Sigma_q$ . Let  $\mathcal{U}_\tau$  be the open neighbourhood of  $\Sigma_q$  which is the union of a finite number of *disjoint* geodesic disks  $D_\tau(p)$  of radius  $\tau \in (0, \tau_0)$ , each centered at a point  $p \in \Sigma_q$ . Let  $\phi_\tau : M \rightarrow [0, 1]$  be a smooth function such that (a)  $\phi_\tau \in C_0^\infty(M \setminus \Sigma_q)$ , (b)  $\phi_\tau \equiv 1$  on  $M \setminus \mathcal{U}_\tau$  and (c) for each  $(i, j) \in \mathbb{N} \times \mathbb{N}$  there exists a constant  $C_{ij} > 0$  such that, for all  $\tau \in (0, \tau_0)$ ,

$$\max_{x \in M} |S^i T^j \phi_\tau(x)| \leq \frac{C_{ij}}{\tau^{i+j}}.$$

It can be proved that, since  $F - f$  vanishes at infinite order at  $\Sigma_q$ ,

$$f + \phi_\tau(F - f) \rightarrow F \quad \text{in } H_q^\infty(M), \quad \text{as } \tau \rightarrow 0^+,$$

which implies, since by construction  $f + \phi_\tau(F - f) \in \Omega_q^0(M)$ , that  $F$  belongs to the closure of  $\Omega_q^0(M)$  in  $H_q^\infty(M)$ .  $\square$

**Definition 3.7.** The space  $\mathcal{S}_q(M) \subset \mathcal{D}(M \setminus \Sigma_q)$  of *q-tempered currents* (introduced in [For02], §6.1) is the dual space of the Fréchet space  $\Omega_q(M)$ . A *homogeneous q-tempered current* of dimension  $d$  (and degree  $2 - d$ ) is a continuous functional on the subspace  $\Omega_q^d(M) \subset \Omega_q(M)$  of homogeneous forms of degree  $d \in \{0, 1, 2\}$ . The space of homogeneous currents of dimension  $d$  (and degree  $2 - d$ ) will be denoted by  $\mathcal{S}_q^d(M)$ .

For any quadratic differential  $q$  on  $M$ , there is a natural operator  $*$ , which maps the space  $\mathcal{D}^0(M \setminus \Sigma_q)$  of currents of dimension 0 and degree 2 on the non-compact manifold  $M \setminus \Sigma_q$  (which is naturally identified with the space of distributions on  $M \setminus \Sigma_q$ ) bijectively onto the space  $\mathcal{D}^2(M \setminus \Sigma_q)$  of currents of dimension 2 and degree 0 on  $M \setminus \Sigma_q$ . The operator

$$* : \mathcal{D}^0(M \setminus \Sigma_q) \rightarrow \mathcal{D}^2(M \setminus \Sigma_q)$$

is defined as follows. Let  $\omega_q$  be the smooth area form associated with the (orientable) quadratic differential  $q$  on  $M$ . It is a standard fact in the theory of currents that any distribution  $U$  on the 2-dimensional surface  $M \setminus \Sigma_q$  can be written as  $U = U^* \omega_q$  for a unique current  $U^*$  of dimension 2 and degree 0. Since  $\omega_q \in \Omega_q^2(M)$ , the map  $*$  extends to a bijective map

$$* : \mathcal{S}_q^0(M) \rightarrow \mathcal{S}_q^2(M).$$

**Definition 3.8.** A distribution  $\mathcal{D} \in \mathcal{D}^2(M \setminus \Sigma_q)$  is *horizontally [vertically] quasi-invariant* if  $S\mathcal{D} = 0$  [ $T\mathcal{D} = 0$ ] in  $\mathcal{D}^2(M \setminus \Sigma_q)$ . A distribution  $\mathcal{D} \in \mathcal{S}_q^2(M)$  is *horizontally [vertically] invariant* if  $S\mathcal{D} = 0$  [ $T\mathcal{D} = 0$ ] in  $\mathcal{S}_q^2(M)$ . The space of horizontally [vertically] quasi-invariant distributions will be denoted by  $\mathcal{I}_q(M \setminus \Sigma_q)$  [ $\mathcal{I}_q(M \setminus \Sigma_q)$ ] and the subspace of horizontally [vertically] invariant distributions will be denoted by  $\mathcal{I}_q(M)$  [ $\mathcal{I}_{-q}(M)$ ].

**Definition 3.9.** For any  $s \in \mathbb{R}^+$ , let

$$(3.25) \quad \begin{aligned} \mathcal{I}_{\pm q}^s(M \setminus \Sigma_q) &:= \mathcal{I}_{\pm q}(M \setminus \Sigma_q) \cap H_q^s(M); \\ \mathcal{I}_{\pm q}^s(M) &:= \mathcal{I}_{\pm q}(M) \cap H_q^s(M). \end{aligned}$$

The subspaces  $\mathcal{I}_{\pm q}^s(M) \subset H_q^{-s}(M)$  of *horizontally [vertically] invariant* distributions can also be defined as follows:

$$(3.26) \quad \begin{aligned} \mathcal{I}_q^s(M) &:= \{\mathcal{D} \in H_q^s(M) \mid S\mathcal{D} = 0 \quad \text{in } H_q^{-s-1}(M)\}; \\ [\mathcal{I}_{-q}^s(M) &:= \{\mathcal{D} \in H_q^s(M) \mid T\mathcal{D} = 0 \quad \text{in } H_q^{-s-1}(M)\}]. \end{aligned}$$

The subspaces of horizontally [vertically] invariant distributions which can be extended to bounded functionals on Friedrichs weighted Sobolev spaces will be denoted by

$$(3.27) \quad \begin{aligned} \bar{\mathcal{I}}_{\pm q}^s(M \setminus \Sigma_q) &:= \mathcal{I}_{\pm q}(M \setminus \Sigma_q) \cap \bar{H}_q^{-s}(M); \\ \bar{\mathcal{I}}_{\pm q}^s(M) &:= \mathcal{I}_{\pm q}(M) \cap \bar{H}_q^{-s}(M). \end{aligned}$$

Let  $\mathcal{V}_q(M)$  be the space of vector fields  $X$  on  $M \setminus \Sigma_q$  such that the contraction  $\iota_X \alpha$  and the Lie derivative  $\mathcal{L}_X \alpha \in \Omega_q(M)$  for all  $\alpha \in \Omega_q(M)$ .

**Definition 3.10.** A current  $C \in \mathcal{D}^1(M \setminus \Sigma_q)$  is *horizontally [vertically] quasi-basic*, that is basic for  $\mathcal{F}_q$  [ $\mathcal{F}_{-q}$ ] in the standard sense on  $M \setminus \Sigma_q$ , if the identities

$$(3.28) \quad \iota_X C = \mathcal{L}_X C = 0$$

hold in  $\mathcal{D}(M \setminus \Sigma_q)$  for all smooth vector fields  $X$  tangent to  $\mathcal{F}_q$  [ $\mathcal{F}_{-q}$ ] with compact support on  $M \setminus \Sigma_q$ . A  $q$ -tempered current  $C \in \mathcal{S}_q^1(M)$  is *horizontally [vertically] basic* if the identities (3.28) holds in  $\mathcal{S}_q(M)$  for all vector fields  $X \in \mathcal{V}_q(M)$ , tangent to  $\mathcal{F}_q$  [ $\mathcal{F}_{-q}$ ] on  $M \setminus \Sigma_q$ . The vector spaces of horizontally [vertically] quasi-basic (real) currents will be denoted by  $\mathcal{B}_q(M \setminus \Sigma_q)$  [ $\mathcal{B}_{-q}(M \setminus \Sigma_q)$ ] and the subspace of horizontally [vertically] basic (real) currents will be denoted by  $\mathcal{B}_q(M)$  [ $\mathcal{B}_{-q}(M)$ ].

**Definition 3.11.** For any  $s \in \mathbb{R}$ , the Friedrichs weighted Sobolev space of 1-currents  $\bar{W}_q^s(M) \subset \mathcal{S}_q(M)$  and the weighted Sobolev space of 1-currents  $W_q^s(M) \subset \mathcal{S}_q(M)$  are defined as follows:

$$(3.29) \quad \begin{aligned} \bar{W}_q^s(M) &:= \{\alpha \in \mathcal{S}_q(M) \mid (\iota_S \alpha, \iota_T \alpha) \in \bar{H}_q^s(M) \times \bar{H}_q^s(M)\}; \\ W_q^s(M) &:= \{\alpha \in \mathcal{S}_q(M) \mid (\iota_S \alpha, \iota_T \alpha) \in H_q^s(M) \times H_q^s(M)\}. \end{aligned}$$

**Definition 3.12.** For any 1-current  $C \in \mathcal{D}(M \setminus \Sigma_q)$ , the *weighted Sobolev order*  $\mathcal{O}_q^W(C)$  and the *Friedrichs weighted Sobolev order*  $\bar{\mathcal{O}}_q^W(C)$  are the real numbers defined as follows:

$$(3.30) \quad \begin{aligned} \mathcal{O}_q^W(C) &:= \inf\{s \in \mathbb{R} \mid \mathcal{D} \in W_q^{-s}(M)\}; \\ \bar{\mathcal{O}}_q^W(C) &:= \inf\{s \in \mathbb{R} \mid C \in \bar{W}_q^{-s}(M)\}. \end{aligned}$$

**Definition 3.13.** For any  $s \in \mathbb{R}$ , let

$$(3.31) \quad \begin{aligned} \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q) &:= \mathcal{B}_{\pm q}(M \setminus \Sigma_q) \cap W_q^s(M); \\ \mathcal{B}_{\pm q}^s(M) &:= \mathcal{B}_{\pm q}(M) \cap W_q^s(M). \end{aligned}$$

The subspaces  $\mathcal{B}_{\pm q}^s(M) \subset W_q^{-s}(M)$  of *horizontally [vertically] basic currents* can also be defined as follows:

$$(3.32) \quad \begin{aligned} \iota_S C = 0 \text{ in } H_q^{-s}(M) \quad \text{and} \quad \mathcal{L}_S C = 0 \text{ in } W_q^{-s-1}(M); \\ [\iota_T C = 0 \text{ in } H_q^{-s}(M) \quad \text{and} \quad \mathcal{L}_T C = 0 \text{ in } W_q^{-s-1}(M)]. \end{aligned}$$

The subspaces of basic currents which can be extended to bounded functionals on Friedrichs weighted Sobolev spaces will be denoted by

$$(3.33) \quad \begin{aligned} \bar{\mathcal{B}}_{\pm q}^s(M \setminus \Sigma_q) &:= \mathcal{B}_{\pm q}(M \setminus \Sigma_q) \cap \bar{W}_q^{-s}(M); \\ \bar{\mathcal{B}}_{\pm q}^s(M) &:= \mathcal{B}_{\pm q}(M) \cap \bar{W}_q^{-s}(M). \end{aligned}$$

According to Lemma 6.5 of [For02], the notions of invariant distributions and basic currents are related (see also Lemma 6.6 in [For02]):

**Lemma 3.14.** A current  $C \in \mathcal{B}_q^s(M \setminus \Sigma_q)$  [ $C \in \mathcal{B}_q^s(M)$ ] if and only if the distribution  $C \wedge \mathfrak{R}(q^{1/2}) \in \mathcal{I}_q^s(M \setminus \Sigma_q)$  [ $C \wedge \mathfrak{R}(q^{1/2}) \in \mathcal{I}_q^s(M)$ ]. A current  $C \in \mathcal{B}_{-q}^s(M \setminus \Sigma_q)$  [ $C \in \mathcal{B}_{-q}^s(M)$ ] if and only if the distribution  $C \wedge \mathfrak{I}(q^{1/2}) \in \mathcal{I}_q^s(M \setminus \Sigma_q)$  [ $C \wedge \mathfrak{I}(q^{1/2}) \in \mathcal{I}_q^s(M)$ ]. In addition, the map

$$(3.34) \quad \begin{aligned} \mathcal{D}_q : C &\rightarrow -C \wedge \mathfrak{R}(q^{1/2}); \\ [\mathcal{D}_{-q} : C &\rightarrow C \wedge \mathfrak{I}(q^{1/2})]; \end{aligned}$$

is a bijection from the space  $\mathcal{B}_q^s(M \setminus \Sigma_q)$  [ $\mathcal{B}_{-q}^s(M \setminus \Sigma_q)$ ] onto the space  $\mathcal{I}_q^s(M \setminus \Sigma_q)$  [ $\mathcal{I}_{-q}^s(M \setminus \Sigma_q)$ ], which maps the subspace  $\mathcal{B}_q^s(M)$  [ $\mathcal{B}_{-q}^s(M)$ ] onto the subspace  $\mathcal{I}_q^s(M)$  [ $\mathcal{I}_{-q}^s(M)$ ]. The map (3.34) also maps the space  $\bar{\mathcal{B}}_q^s(M)$  [ $\bar{\mathcal{B}}_{-q}^s(M)$ ] onto  $\bar{\mathcal{I}}_q^s(M)$  [ $\bar{\mathcal{I}}_{-q}^s(M)$ ].

**3.3. Basic cohomology.** Let  $\mathcal{Z}(M \setminus \Sigma) \subset \mathcal{D}^1(M \setminus \Sigma)$  denote the subspace of all (real) closed currents, that is, the space of all (real) de Rham currents  $C \in \mathcal{D}^1(M \setminus \Sigma)$  such that the exterior derivative  $dC = 0$  in  $\mathcal{D}(M \setminus \Sigma)$ . Let  $\mathcal{Z}_q(M) \subset \mathcal{S}_q^1(M)$  be the subspace of all (real) closed  $q$ -tempered currents, that is, the space of all  $q$ -tempered (real) currents  $C$  such that  $dC = 0$  in

$\mathcal{S}_q(M)$ . It was proved in [For02], Lemma 6.2, that the natural de Rham cohomology map

$$(3.35) \quad j_q : \mathcal{Z}(M \setminus \Sigma_q) \rightarrow H^1(M \setminus \Sigma_q, \mathbb{R})$$

has the property that the subspace of closed  $q$ -tempered currents is mapped onto the absolute real cohomology of the surface, that is,

$$(3.36) \quad j_q : \mathcal{Z}_q(M) \rightarrow H^1(M, \mathbb{R}) \subset H^1(M \setminus \Sigma_q, \mathbb{R}).$$

It was also proved in [For02], Lemma 6.2', that quasi-basic and basic currents are closed, in the sense that the following inclusions hold:

$$(3.37) \quad \begin{aligned} \mathcal{B}_{\pm q}(M \setminus \Sigma_q) &\subset \mathcal{Z}(M \setminus \Sigma_q), \\ \mathcal{B}_{\pm q}(M) &\subset \mathcal{Z}_q(M). \end{aligned}$$

The images of the restrictions of the natural cohomology map to the various spaces of basic currents are called the *horizontal [vertical] basic cohomologies*, namely the spaces

$$(3.38) \quad \begin{aligned} H_{\pm q}^1(M \setminus \Sigma_q, \mathbb{R}) &:= j_q(\mathcal{B}_{\pm q}(M \setminus \Sigma_q)) \subset H^1(M \setminus \Sigma_q, \mathbb{R}); \\ H_{\pm q}^{1,s}(M \setminus \Sigma_q, \mathbb{R}) &:= j_q(\mathcal{B}_{\pm q}^s(M \setminus \Sigma_q)) \subset H_{\pm q}^1(M \setminus \Sigma_q, \mathbb{R}); \\ H_{\pm q}^1(M, \mathbb{R}) &:= j_q(\mathcal{B}_{\pm q}(M)) \subset H^1(M, \mathbb{R}); \\ H_{\pm q}^{1,s}(M, \mathbb{R}) &:= j_q(\mathcal{B}_{\pm q}^s(M)) \subset H_{\pm q}^1(M, \mathbb{R}). \end{aligned}$$

Following [For02], Theorem 7.1, we give below a description of the horizontal [vertical] basic cohomologies for the orientable quadratic differential  $q_\theta$ , for any orientable holomorphic quadratic differential  $q$  on  $M$  and for almost all  $\theta \in S^1$ . The result we obtain below is stronger than Theorem 7.1 of [For02] since it requires weaker Sobolev regularity assumptions.

(Absolute) real cohomology classes on  $M$  can be represented in terms of meromorphic (or anti-meromorphic) functions in  $L_q^2(M)$  (see [For02], §2). In fact, by the Hodge theory on Riemann surfaces [FK92], III.2, all real cohomology classes can be represented as the real (or imaginary) part of a holomorphic (or anti-holomorphic) differential on  $M$ . In turn, any orientable holomorphic quadratic differential induces an isomorphism between the space  $\text{Hol}^+(M)$  [ $\text{Hol}^-(M)$ ] of holomorphic [anti-holomorphic] differentials and the space of square-integrable meromorphic [anti-meromorphic] functions. Let  $\mathcal{M}_q^+$  [ $\mathcal{M}_q^-$ ] be the space of meromorphic [anti-meromorphic] functions on  $M$  which belong to the Hilbert space  $L_q^2(M)$  (see Proposition 2.4). Such spaces can be characterized as the spaces of all meromorphic [anti-meromorphic] functions with poles at  $\Sigma_q = \{q = 0\}$  of orders bounded in terms of the multiplicity of the points  $p \in \Sigma_q$  as zeroes of the quadratic differential  $q$ . In fact, if  $p \in \Sigma_q$  is a zero of  $q$  of order  $2m$ , that  $p$  is a pole of order at most  $m$  for any  $m^\pm \in \mathcal{M}_q^\pm$ .



Let  $q^{1/2}$  be a holomorphic square root of  $q$  on  $M$ . Holomorphic [anti-holomorphic] differentials  $h^+$  [ $h^-$ ] on  $M$  can be written in terms of meromorphic [anti-meromorphic] functions in  $L_q^2(M)$  as follows:

$$(3.39) \quad \begin{aligned} h^+ &:= m^+ q^{1/2}, & m^+ &\in \mathcal{M}_q^+; \\ h^- &:= m^- \bar{q}^{1/2}, & m^- &\in \mathcal{M}_q^-. \end{aligned}$$

The following *representations of real cohomology classes* therefore hold:

$$(3.40) \quad \begin{aligned} c \in H^1(M, \mathbb{R}) &\iff c = [\Re(m^+ q^{1/2})], & m^+ &\in \mathcal{M}_q^+; \\ c \in H^1(M, \mathbb{R}) &\iff c = [\Re(m^- \bar{q}^{1/2})], & m^- &\in \mathcal{M}_q^-. \end{aligned}$$

The maps  $c_q^\pm : \mathcal{M}_q^\pm \rightarrow H^1(M, \mathbb{R})$  given by the representations (3.40) are bijective and it is in fact *isometric* if the spaces  $\mathcal{M}_q^\pm$  are endowed with the euclidean structure induced by  $L_q^2(M)$  and  $H^1(M, \mathbb{R})$  with the *Hodge product* relative to the complex structure of the Riemann surface  $M$ . In fact, the Hodge norm  $\|c\|_H^2$  of a cohomology class  $c \in H^1(M, \mathbb{R})$  is defined as follows:

$$(3.41) \quad \|c\|_H^2 := \frac{i}{2} \int_M h^\pm \wedge \overline{h^\pm} \quad \text{if } c = [\Re(h^\pm)], \quad h^\pm \in \text{Hol}^\pm(M).$$

We remark that the Hodge norm is defined in terms of the complex structure of the Riemann surface  $M$  (carrying a holomorphic quadratic differential  $q \in Q(M)$ ) but does not depend on the quadratic differential. If  $q \in Q(M)$  is any orientable quadratic differential on  $M$ , by the representation (3.40), we can also write:

$$(3.42) \quad \begin{aligned} \|c_q^+(m^+)\|_H^2 &:= \int_M |m^+|^2 \omega_q, & \text{for all } m^+ &\in \mathcal{M}_q^+; \\ \|c_q^-(m^-)\|_H^2 &:= \int_M |m^-|^2 \omega_q, & \text{for all } m^- &\in \mathcal{M}_q^-. \end{aligned}$$

The representation (3.39)-(3.40) can be extended to the punctured cohomology  $H^1(M \setminus \Sigma_q, \mathbb{R})$  as follows. For any finite set  $\Sigma \subset M$ , let  $\text{Hol}^+(M \setminus \Sigma)$  [ $\text{Hol}^-(M \setminus \Sigma)$ ] be the space of meromorphic [anti-meromorphic] differentials with at most simple poles at  $\Sigma$ . By Riemann surface theory, any real cohomology class  $c \in H^1(M \setminus \Sigma, \mathbb{R})$  can be represented as the real (or imaginary) part of a differential  $h^+ \in \text{Hol}^+(M \setminus \Sigma)$  or  $h^- \in \text{Hol}^-(M \setminus \Sigma)$ . Let  $\Sigma \subset M$  be a finite set and let  $\mathcal{M}^+(\Sigma)$  [ $\mathcal{M}^-(\Sigma)$ ] be the space of all meromorphic [anti-meromorphic] functions which are holomorphic [anti-holomorphic] on  $M \setminus \Sigma$ . The spaces  $\mathcal{M}^\pm(\Sigma)$  can be identified with a subspace of the distributional space  $\mathcal{D}^2(M \setminus \Sigma)$ . In fact, if  $q$  is any orientable holomorphic quadratic differential on  $M$ , the spaces  $\mathcal{M}^\pm(\Sigma_q)$  identify with subspaces of the space  $\mathcal{S}_q^2(M)$  of  $q$ -tempered distributions. The distribution determined by a function  $\phi \in \mathcal{M}^+(\Sigma_q)$  or  $\mathcal{M}^-(\Sigma_q)$  is defined by integration

(in the standard way) as a linear functional on  $C_0^\infty(M \setminus \Sigma_q)$ , which can be extended to the space  $\Omega_q^0(M)$  as follows:

$$\phi(v) := \text{PV} \int_M \phi v \omega_q, \quad \text{for all } v \in \Omega_q^0(M).$$

The Sobolev regularity of a distributions  $\phi \in \mathcal{M}^\pm(\Sigma_q)$  depend on the order of its poles. In fact, by Theorem 2.22 we have the following:

**Lemma 3.15.** *Let  $\phi \in \mathcal{M}^+(\Sigma_q)$  [  $\phi \in \mathcal{M}^-(\Sigma_q)$  ] be a meromorphic [anti-meromorphic] function with poles at  $\Sigma_q$ . For any  $s \in \mathbb{R}$ , the associated distribution  $\phi \in H_q^{-s}(M)$  if at every  $p \in \Sigma_q$  of order  $2m_p$  the function  $\phi$  has a pole of order  $< (m_p + 1)(s + 1)$ .*

We introduce the following notation: for all  $s > 0$ ,

$$(3.43) \quad \mathcal{M}_s^\pm(\Sigma_q) := \mathcal{M}^\pm(\Sigma_q) \cap H_q^{-s}(M).$$

There exist natural maps  $\phi_q^\pm : \text{Hol}^\pm(M \setminus \Sigma_q) \rightarrow \mathcal{M}^\pm(\Sigma_q)$  defined as follows: for all  $h^\pm \in \text{Hol}^\pm(M \setminus \Sigma_q)$ ,

$$\phi_q^+(h^+) = h^+ / q^{1/2} \quad [\phi_q^-(h^-) = h^- / \bar{q}^{1/2}].$$

By Lemma 3.15 the range of the maps  $\phi_q^\pm$  is contained in the weighted Sobolev space  $H_q^{-s}(M)$  for all  $s > 0$ , hence there are well-defined maps

$$(3.44) \quad \phi_q^\pm : \text{Hol}^\pm(M \setminus \Sigma_q) \rightarrow \mathcal{M}_s^\pm(\Sigma_q) \quad \text{for all } s > 0.$$

The maps (3.44) are clearly injective and by Corollary 2.23 there exists  $s_q > 0$  such that, for any  $s \in (0, s_q)$ , they are also surjective. Let

$$(3.45) \quad \mathcal{M}_*^\pm(\Sigma_q) = \bigcap_{s>0} \mathcal{M}_s^\pm(\Sigma_q) = \mathcal{M}_s^\pm(\Sigma_q), \quad \text{for any } s \in (0, s_q).$$

The representation (3.40) of the absolute real cohomology generalizes to the punctured real cohomology as follows.

$$(3.46) \quad \begin{aligned} c \in H^1(M \setminus \Sigma_q, \mathbb{R}) &\iff c = [\Re(m^+ q^{1/2})], \quad m^+ \in \mathcal{M}_*^+(\Sigma_q); \\ c \in H^1(M \setminus \Sigma_q, \mathbb{R}) &\iff c = [\Re(m^- \bar{q}^{1/2})], \quad m^- \in \mathcal{M}_*^-(\Sigma_q). \end{aligned}$$

The following lemma, proved in [For02], Lemma 7.6, for weighted Sobolev spaces with integer exponent, holds:

**Lemma 3.16.** *Let  $s \in \mathbb{R}^+$ . Let  $C \in W_q^{-s}(M)$  be any real current of dimension (and degree) equal to 1, closed in the space  $\mathcal{D}(M \setminus \Sigma_q)$  of currents on  $M \setminus \Sigma_q$ . There exists a distribution  $U \in H_q^{-s+1}(M)$  and a meromorphic differential  $h^+ \in \text{Hol}^+(M \setminus \Sigma_q)$  such that*

$$(3.47) \quad dU^* = \Re(h^+) - C \quad \text{in } W_q^{-s}(M).$$

If  $C$  is closed in the space  $\mathcal{S}_q(M)$  of  $q$ -tempered currents there exists a distribution  $U \in H_q^{-s+1}(M)$  and a holomorphic differential  $h^+ \in \text{Hol}^+(M)$  such that the identity (3.47) holds.

The argument given in [For02], Lemma 7.6, in the case of integer order  $s \in \mathbb{N}$  extends the general case of order  $s \in \mathbb{R}^+$ . In fact, it follows from the distributional identity (3.47) in  $\mathcal{S}_q(M)$  that the current  $U^* \in H_q^{-s+1}(M)$  if and only if the current  $C \in W_q^{-s}(M)$ , for any  $s \in \mathbb{R}^+$ . Hence, Lemma 3.16 follows immediately from [For02], Lemma 7.6.

The construction of basic currents (or, equivalently, of invariant distributions) is based on the following method.

**Lemma 3.17.** *Let  $q$  be an orientable holomorphic quadratic differential on a Riemann surface  $M$ . Let  $m^+ \in \mathcal{M}_s^+(\Sigma_q)$  be a meromorphic function with poles at  $\Sigma_q \subset M$ . A distribution  $U \in H_q^{-s+1}(M)$  is a (distributional) solution in  $\mathcal{D}(M \setminus \Sigma_q)$  of the cohomological equation*

$$(3.48) \quad SU = \Re(m^+) [TU = -\Im(m^+)] \quad \text{in } \mathcal{D}(M \setminus \Sigma_q),$$

*if and only if the current  $C \in W_q^{-s}(M)$  uniquely determined by the identity*

$$(3.49) \quad dU^* = \Re(m^+ q^{1/2}) + C$$

*is horizontally [vertically] quasi-basic. If  $\Re(m^+ q^{1/2}) \in \text{Hol}^+(M)$ , the distribution  $U \in H_q^{-s+1}(M)$  is a solution of the cohomological equation (3.48) in the space  $\mathcal{S}_q(M)$  of  $q$ -tempered currents if and only if the current  $C \in W_q^{-s}(M)$  uniquely determined by formula (3.49) is horizontally [vertically] basic.*

*Proof.* If formula (3.49) holds in  $\mathcal{D}(M \setminus \Sigma_q)$ , then  $C$  is closed in  $\mathcal{D}(M \setminus \Sigma_q)$ . If the differential  $\Re(m^+ q^{1/2})$  is holomorphic and formula (3.49) holds in  $\mathcal{S}_q(M)$ , then  $C$  is closed in  $\mathcal{S}_q(M)$ . The standard formula for the Lie derivative of a current,

$$(3.50) \quad \mathcal{L}_X C = \iota_X dC + d\iota_X C = 0,$$

holds in  $\mathcal{D}(M \setminus \Sigma_q)$  for any vector field  $X$  with compact support contained in  $M \setminus \Sigma_q$  and it holds in  $\mathcal{S}_q(M)$  for any vector field  $X \in \mathcal{V}_q(M)$ . It follows that a current  $C \in \mathcal{D}(M \setminus \Sigma_q)$  is horizontally [vertically] quasi-basic if and only if it is closed and  $\iota_S C = 0$  [ $\iota_T C = 0$ ] in  $\mathcal{D}(M \setminus \Sigma_q)$  and it is horizontally [vertically] basic if and only if  $C \in \mathcal{S}_q(M)$  is closed and  $\iota_S C = 0$  [ $\iota_T C = 0$ ] in  $\mathcal{S}_q(M)$ . The distribution  $U \in H_q^{-s}(M)$  in formula (3.49) is a solution of the cohomological equation (3.48) in  $\mathcal{D}(M \setminus \Sigma_q)$  or  $\mathcal{S}_q(M)$  if and only if  $\iota_S C = 0$  [ $\iota_T C = 0$ ] in  $\mathcal{D}(M \setminus \Sigma_q)$  or  $\mathcal{S}_q(M)$  respectively. As a consequence, the lemma is proved.  $\square$

Let  $q$  be an orientable holomorphic quadratic differential on a Riemann surface  $M$  (of genus  $g \geq 1$ ). Let  $\Pi_{\pm q}(M \setminus \Sigma_q, \mathbb{R}) \subset H^1(M \setminus \Sigma_q, \mathbb{R})$  be the codimension 1 subspaces defined as follows:

$$(3.51) \quad \begin{aligned} \Pi_q^1(M \setminus \Sigma_q, \mathbb{R}) &:= \{c \in H^1(M \setminus \Sigma_q, \mathbb{R}) \mid c \wedge [\Im(q^{1/2})] = 0\}; \\ \Pi_{-q}^1(M \setminus \Sigma_q, \mathbb{R}) &:= \{c \in H^1(M \setminus \Sigma_q, \mathbb{R}) \mid c \wedge [\Re(q^{1/2})] = 0\}. \end{aligned}$$

Since the absolute cohomology can be regarded as a subspace of the punctured cohomology it also is possible to define the subspaces

$$(3.52) \quad \begin{aligned} \Pi_q^1(M, \mathbb{R}) &:= \Pi_q^1(M \setminus \Sigma_q, \mathbb{R}) \cap H^1(M, \mathbb{R}); \\ \Pi_{-q}^1(M, \mathbb{R}) &:= \Pi_{-q}^1(M \setminus \Sigma_q, \mathbb{R}) \cap H^1(M, \mathbb{R}). \end{aligned}$$

**Theorem 3.18.** *For any  $s > 3$  there exists a full measure set  $\mathcal{F}_s \subset S^1$  such that the following holds. For any  $\theta \in \mathcal{F}_s$ , the following inclusions hold*

$$(3.53) \quad \begin{aligned} \Pi_{\pm q\theta}^1(M \setminus \Sigma_q, \mathbb{R}) &\subset H_{\pm q\theta}^{1,s}(M \setminus \Sigma_q, \mathbb{R}), \\ \Pi_{\pm q\theta}^1(M, \mathbb{R}) &\subset H_{\pm q\theta}^{1,s}(M, \mathbb{R}). \end{aligned}$$

*Proof.* Let  $m^+ \in \mathcal{M}_{-1}^+(\Sigma_q)$  be any meromorphic function such that the induced distribution  $\text{PV}(m^+) \in H_q^{-1}(M)$ . A computation shows that, for all  $\theta \in S^1$ ,

$$(3.54) \quad \text{PV} \int_M \Re(m^+) \omega_q = 0 \iff [\Re(m^+ q_\theta^{1/2})] \in \Pi_{q\theta}^1(M \setminus \Sigma_q, \mathbb{R}).$$

Under the zero-average condition (3.54), by Theorem 3.2 for any  $r > 2$  there exists a full measure set  $\mathcal{F}_r(m^+) \subset S^1$  such that, for all  $\theta \in S^1$ , the cohomological equation  $S_\theta U = \Re(m^+)$  has a distributional solution  $U_\theta(m^+) \in \bar{H}_q^{-r}(M)$ . Let  $U_\theta^*(m^+)$  be the current of dimension 2 corresponding to the distribution  $U_\theta(m^+)$ . Let  $C_\theta(m^+) \in W_q^{-r-1}(M)$  be the 1-dimensional current determined by the identity

$$(3.55) \quad dU_\theta^*(m^+) = \Re(m^+ q_\theta^{1/2}) - C_\theta(m^+).$$

By Lemma 3.17, we have thus proved that, for all meromorphic functions  $m^+ \in \mathcal{M}_{-1}^+(\Sigma_q)$  and for all  $\theta \in \mathcal{F}_r(m^+)$ , there exists a quasi-basic current  $C_\theta(m^+) \in \mathcal{B}_{q\theta}^{r+1}(M \setminus \Sigma_q)$  such that

$$[C_\theta(m^+)] = [\Re(m^+ q_\theta^{1/2})] \in \Pi_{q\theta}^1(M \setminus \Sigma_q, \mathbb{R});$$

in addition, whenever  $m^+ \in \mathcal{M}_q^+$ , the current  $C_\theta(m^+) \in \mathcal{B}_{q\theta}^{r+1}(M)$  is basic and has a cohomology class

$$[C_\theta(m^+)] = [\Re(m^+ q_\theta^{1/2})] \in \Pi_{q\theta}^1(M, \mathbb{R}).$$

Let  $\sigma \in \mathbb{N}$  be the cardinality of the set  $\Sigma_q \subset M$  and let

$$\{m_1^+, \dots, m_{2g-1}^+, \dots, m_{2g+\sigma-1}^+\}$$

be a basis (over  $\mathbb{R}$ ) of the real subspace of  $\mathcal{M}_*^+(\Sigma_q)$  defined by the zero-average condition (3.54). For any  $s > 3$ , let

$$\mathcal{F}_s^+ := \bigcap_{i=1}^{2g+\sigma-1} \mathcal{F}_{s-1}(m_i^+).$$

Clearly the set  $\mathcal{F}_s^+$  has full Lebesgue measure. We claim that for all  $\theta \in \mathcal{F}_s^+$  the following inclusions hold:

$$(3.56) \quad \begin{aligned} \Pi_{q\theta}^1(M \setminus \Sigma_q, \mathbb{R}) &\subset H_{q\theta}^{1,s}(M \setminus \Sigma_q, \mathbb{R}), \\ \Pi_{q\theta}^1(M, \mathbb{R}) &\subset H_{q\theta}^{1,s}(M \setminus \mathbb{R}). \end{aligned}$$

The claim is proved as follows. For any  $c \in H^1(M \setminus \Sigma_q, \mathbb{R})$  there exists a unique meromorphic function  $m^+ \in \mathcal{M}_*^+(\Sigma_q)$  such that  $c = [\Re(m^+ q_\theta^{1/2})]$ . The function  $m^+ \in \mathcal{M}_q^+$  for any  $c \in H^1(M, \mathbb{R})$ . If  $c \in \Pi_{q\theta}^1(M, \mathbb{R})$ , the distribution  $\Re(m^+)$  vanishes on constant functions as in (3.54), hence for all  $\theta \in \mathcal{F}_s^+$ , there exists a solution  $U \in H_q^{-s+1}(M)$  of the cohomological equation  $S_\theta U = \Re(m^+)$ . The current  $C \in \mathcal{B}_{q\theta}^s(M \setminus \Sigma_q)$  such that  $[C] = c \in \Pi_{q\theta}^1(M \setminus \Sigma_q, \mathbb{R})$  is then given by the identity (3.55). By the above discussion the current  $C \in \mathcal{B}_{q\theta}^s(M)$  for all  $c \in \Pi_{q\theta}^1(M, \mathbb{R})$ .

By a similar argument it is possible to construct a full measure set  $\mathcal{F}_s^-$  such that, for all  $\theta \in \mathcal{F}_s^-$ , the following inclusions hold:

$$(3.57) \quad \begin{aligned} \Pi_{-q\theta}^1(M \setminus \Sigma_q, \mathbb{R}) &\subset H_{-q\theta}^{1,s}(M \setminus \Sigma_q, \mathbb{R}), \\ \Pi_{-q\theta}^1(M, \mathbb{R}) &\subset H_{-q\theta}^{1,s}(M, \mathbb{R}). \end{aligned}$$

Thus the set  $\mathcal{F}_s := \mathcal{F}_s^+ \cap \mathcal{F}_s^-$  has the required properties since it has full measure and the inclusions (3.53) hold.  $\square$

By Lemma 3.14 and Theorem 3.18 the following holds:

**Corollary 3.19.** *For any  $s > 3$  there exists a full measure set  $\mathcal{F}_s \subset S^1$  such that, for all  $\theta \in \mathcal{F}_s$ , the spaces  $\mathcal{J}_{\pm q\theta}^s(M) \subset H_q^{-s}(M)$  of horizontally or vertically quasi-invariant distributions have dimension at least  $2g + \sigma - 1$  and the spaces  $\mathcal{J}_{\pm q\theta}^s(M) \subset H_q^{-s}(M)$  of horizontally or vertically invariant distributions have dimension at least  $2g - 1$ .*

**Corollary 3.20.** *For any  $s > 3$  and for almost all  $\theta \in S^1$ ,*

$$(3.58) \quad H_{\pm q\theta}^{1,s}(M, \mathbb{R}) = \Pi_{\pm q\theta}^1(M, \mathbb{R}).$$

*For any  $s > 4$  and for almost all  $\theta \in S^1$ ,*

$$(3.59) \quad H_{\pm q\theta}^{1,s}(M \setminus \Sigma_q, \mathbb{R}) = H^1(M \setminus \Sigma_q, \mathbb{R}).$$

*Proof.* The inclusions  $H_{\pm q}^{1,s}(M, \mathbb{R}) \subset \Pi_{\pm q}^1(M, \mathbb{R})$  hold for any orientable quadratic differential  $q$  on  $M$  and for any  $s > 0$ . In fact,

$$(3.60) \quad \begin{aligned} [C \wedge \Im(q^{1/2})](1) &= \iota_S C(\omega_q) = 0, & \text{if } C \in \mathcal{B}_q(M); \\ [C \wedge \Re(q^{1/2})](1) &= -\iota_T C(\omega_q) = 0, & \text{if } C \in \mathcal{B}_{-q}(M). \end{aligned}$$

Thus, the identity (3.58) follows immediately from Theorem 3.18.

By Theorem 3.18, in order to prove the identity (3.59) it is enough to prove that for almost all  $\theta \in S^1$  the cohomology class  $[\Re(q_\theta)] \in \mathcal{B}_{q_\theta}(M \setminus \Sigma_q)$  and the cohomology class  $[\Im(q_\theta)] \in \mathcal{B}_{-q_\theta}(M \setminus \Sigma_q)$ . By Lemma 3.17 the argument is therefore reduced to the construction, for any  $s > 3$  and for almost all  $\theta \in S^1$ , of a solution  $U \in H_q^{-s}(M)$  of the cohomological equation  $S_\theta U = 1$  [ $T_\theta U = 1$ ] in  $\mathcal{D}(M \setminus \Sigma_q)$ . Such a construction can be carried out as follows. Let  $\delta_p$  be the Dirac mass at any point  $p \in \Sigma_q$ . The distribution  $F = 1 - \delta_p \in H^{-s}(M) \subset H_q^{-s}(M)$  for any  $s > 1$ . We claim that for any  $s > 3$  and for almost all  $\theta \in S^1$  there exists a distributional solution  $U \in H_q^{-s}(M)$  of the cohomological equation  $S_\theta U = F$  [ $T_\theta U = F$ ] in  $H_q^{-s-1}(M)$ . It follows that  $U$  is a solution of the cohomological equation  $S_\theta U = 1$  [ $T_\theta U = 1$ ] in  $\mathcal{D}(M \setminus \Sigma_q)$ . The above claim is proved as follows. By [For97], Prop. 4.6, or [For02], Lemma 7.3, since  $F \in H_q^{-2}(M)$  vanishes on constant functions, there exists a distribution  $f \in H_q^{-1}(M)$  such that  $\partial_q^+ f = F$  (as well as a distribution  $f' \in H_q^{-1}(M)$  such that  $\partial_q^- f' = F$ ). By Theorem 3.2, for almost all  $\theta \in S^1$  and for all  $s > 2$ , there exists a solution  $u \in H_q^{-s}(M)$  of the equation  $S_\theta u = f$  [ $T_\theta u = f$ ]. The distribution  $U := \partial_q^+ u \in H_q^{-s}(M)$  for any  $s > 3$  and solves the cohomological equation  $S_\theta U = F$  [ $T_\theta U = F$ ] in  $H_q^{-s-1}(M)$ .  $\square$

The structure of the space of basic currents with vanishing cohomology class with respect to the filtration induced by weighted Sobolev spaces with integer exponent was described in [For02], §7. We extend below the results of [For02] to fractional weighted Sobolev spaces.

Let  $\delta_{\pm q} : \mathcal{B}_{\pm q}(M \setminus \Sigma_q) \rightarrow \mathcal{B}_{\pm q}(M \setminus \Sigma_q)$  be the linear maps defined as follows (see [For02], formula (7.18')):

$$(3.61) \quad \begin{aligned} \delta_q(C) &:= -d(C \wedge \Re(q^{1/2}))^*, & \text{for } C \in \mathcal{B}_q(M \setminus \Sigma_q); \\ \delta_{-q}(C) &:= d(C \wedge \Im(q^{1/2}))^*, & \text{for } C \in \mathcal{B}_{-q}(M \setminus \Sigma_q). \end{aligned}$$

It can be proved by Lemma 3.14 and by the definition of the weighted Sobolev spaces  $H_q^s(M)$  and  $W_q^s(M)$  that the above formulas (3.61) define, for all  $s \in \mathbb{R}^+$ , bounded linear maps

$$(3.62) \quad \begin{aligned} \delta_{\pm q}^s &: \mathcal{B}_{\pm q}^{s-1}(M \setminus \Sigma_q) \rightarrow \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q); \\ \delta_{\pm q}^s &: \mathcal{B}_{\pm q}^{s-1}(M) \rightarrow \mathcal{B}_{\pm q}^s(M). \end{aligned}$$

We remark that a similar statement is false in general for the Friedrichs Sobolev spaces of currents. The following result extends Theorem 7.7 of [For02] to fractional weighted Sobolev spaces.

**Theorem 3.21.** *For all  $s \in \mathbb{R}^+$  there exist exact sequences*

$$(3.63) \quad \begin{aligned} 0 \rightarrow \mathbb{R} \rightarrow \mathcal{B}_{\pm q}^{s-1}(M \setminus \Sigma_q) \xrightarrow{\delta_{\pm q}^s} \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q) \xrightarrow{j_q} H^1(M \setminus \Sigma_q, \mathbb{R}) ; \\ 0 \rightarrow \mathbb{R} \rightarrow \mathcal{B}_{\pm q}^{s-1}(M) \xrightarrow{\delta_{\pm q}^s} \mathcal{B}_{\pm q}^s(M) \xrightarrow{j_q} H^1(M, \mathbb{R}) . \end{aligned}$$

*Proof.* The maps  $i_{\pm q} : \mathbb{R} \rightarrow \mathcal{B}_{\pm q}^{s-1}(M) \subset \mathcal{B}_{\pm q}^{s-1}(M \setminus \Sigma_q)$  defined as

$$i_q(\tau) = \tau \eta_S \quad \text{and} \quad i_{-q}(\tau) = \tau \eta_T, \quad \text{for all } \tau \in \mathbb{R},$$

are clearly injective and the kernels  $\text{Ker}(\delta_{\pm q}^s) = i_{\pm q}(\mathbb{R})$ , for all  $s \in \mathbb{R}^+$ . In fact, if a current  $C \in \text{Ker}(\delta_{+q}^s)$ , then the distribution  $(C \wedge \mathfrak{R}(q^{1/2}))^* \in \mathbb{R}$ , hence  $C \wedge \mathfrak{R}(q^{1/2}) \in \mathbb{R} \cdot \omega_q$  and  $C \in \mathbb{R} \cdot \mathfrak{R}(q^{1/2}) + \mathbb{R} \cdot \mathfrak{S}(q^{1/2})$ . It follows that  $C \in \mathbb{R} \cdot \mathfrak{S}(q^{1/2})$  since  $C \in \mathcal{B}_q(M \setminus \Sigma_q)$ . Similarly, if the current  $C \in \text{Ker}(\delta_{-q}^s)$ , it follows that  $C \wedge \mathfrak{S}(q^{1/2}) \in \mathbb{R} \cdot \omega_q$  and  $C \in \mathbb{R} \cdot \mathfrak{R}(q^{1/2})$  since  $C \in \mathcal{B}_{-q}(M \setminus \Sigma_q)$ . This proves the inclusions  $\text{Ker}(\delta_{\pm q}^s) \subset i_{\pm q}(\mathbb{R})$ . The opposite inclusions are immediate.

By Lemma 3.16 a current  $C \in \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q)$  has zero cohomology class, that is, it is in the kernel of the cohomology map, if and only if there exists a distribution  $U_C \in H_q^{-s+1}(M)$  such that  $dU_C^* = C$  in  $\mathcal{D}(M \setminus \Sigma_q)$  and this identity holds in  $\mathcal{S}_q(M)$ , hence in  $W_q^{-s}(M)$ , if  $C \in \mathcal{B}_{\pm q}^s(M)$ . It is immediate to verify that  $U_C \in \mathcal{J}_{\pm q}^{s-1}(M \setminus \Sigma_q)$  if and only if  $C \in \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q)$  and that  $U_C \in \mathcal{J}_{\pm q}^{s-1}(M)$  if and only if  $C \in \mathcal{B}_{\pm q}^s(M)$ . By Lemma 3.14 we have thus proved that the map  $j_q : \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q) \rightarrow H^1(M \setminus \Sigma_q, \mathbb{R})$  has kernel equal to the range  $\delta_{\pm q}^s(\mathcal{B}_{\pm q}^{s-1}(M))$  and that the map  $j_q : \mathcal{B}_{\pm q}^s(M) \rightarrow H^1(M, \mathbb{R})$  has kernel equal to the range  $\delta_{\pm q}^s(\mathcal{B}_{\pm q}^{s-1}(M))$ .  $\square$

Finer results on invariant distributions and on smooth solutions of the cohomological equation for directional flows can be obtained by combining the results of this section with the renormalization method based on the Teichmüller flow and related cocycles, such as the Kontsevich-Zorich cocycle. Our goal is to improve upon the results of Marmi, Moussa and Yoccoz [MMY05] who have studied the cohomological equation for interval exchange transformations solely by methods based on the renormalization dynamics (the Rauzy-Veech-Zorich induction).

#### 4. COCYCLES OVER THE TEICHMÜLLER FLOW

**4.1. The Kontsevich-Zorich cocycle.** The Kontsevich-Zorich cocycle is a multiplicative cocycle over the Teichmüller geodesic flow on the moduli

space of (orientable) holomorphic quadratic differentials on compact Riemann surfaces. This cocycle appears in the study of the dynamics of interval exchange transformations and of (translation) flows on surfaces for which it represents a renormalization dynamics and of the Teichmüller flow itself. In fact, the study tangent cocycle of the Teichmüller flow can be reduced to that of the Kontsevich-Zorich cocycle.

Let  $T_g$  and  $Q_g$  be respectively the *Teichmüller spaces* of complex (conformal) structures and of holomorphic quadratic differentials on a surface of genus  $g \geq 1$ . We recall that the spaces  $T_g$  and  $Q_g$  can be described as follows. Let  $\text{Diff}_0^+(M)$  is the group of orientation preserving diffeomorphisms of the surface  $M$  which are isotopic to the identity (equivalently, it is the connected component of the identity in the Lie group of all orientation preserving diffeomorphisms of  $M$ ). By definition

$$(4.1) \quad \begin{aligned} T_g &:= \{ \text{complex (conformal) structures} \} / \text{Diff}_0^+(M) , \\ Q_g &:= \{ \text{holomorphic quadratic differentials} \} / \text{Diff}_0^+(M) . \end{aligned}$$

A theorem of L. Ahlfors, L. Bers and S. Wolpert states that  $T_g$  has a complex structure holomorphically equivalent to that of a Stein (strongly pseudoconvex) domain in  $\mathbb{C}^{3g-3}$  [Ber74], §6, or [Nag88], Chap. 3, 4 and Appendix §6. The space  $Q_g$  of holomorphic quadratic differentials is a complex vector bundle over  $T_g$  which can be identified to the cotangent bundle of  $T_g$ . Let  $\Gamma_g := \text{Diff}^+(M) / \text{Diff}_0^+(M)$  be the *mapping class group* and let  $R_g, \mathcal{M}_g$  be respectively the *moduli spaces* of complex (conformal) structures and of holomorphic quadratic differentials on a surface of genus  $g \geq 1$ . The spaces  $R_g$  and  $\mathcal{M}_g$  can be described as the quotient spaces:

$$(4.2) \quad R_g := T_g / \Gamma_g , \quad \mathcal{M}_g := Q_g / \Gamma_g ,$$

In case  $g = 1$ , the Teichmüller space  $T_1$  of elliptic curves (complex structures on  $T^2$ ) is isomorphic to the upper half plane  $\mathbb{C}^+$  and the Teichmüller space  $Q_1$  of holomorphic quadratic differentials on elliptic curves is a complex line bundle over  $T_1$  (see [Nag88], Ex. 2.1.8). The mapping class group can be identified with the lattice  $SL(2, \mathbb{Z})$  which acts on the upper half plane  $\mathbb{C}^+$  in the standard way. The moduli space  $R_1 := \mathbb{C}^+ / SL(2, \mathbb{Z})$  is a non-compact finite volume surface with constant negative curvature, called the *modular surface*. The moduli space  $\mathcal{M}_1$  can be identified to the cotangent bundle of the modular surface.

The *Teichmüller (geodesic) flow* on  $\mathcal{M}_g$  can be defined as the geodesic flow for a natural metric on  $R_g$  called the *Teichmüller metric*. Such a metric measures the amount of *quasi-conformal distortion* between two different (equivalent classes of) complex structures in  $R_g$ . In the higher genus case, the Teichmüller metric is not Riemannian, but only *Finsler* (that is,



the norm on each tangent space does not come from an euclidean product) and, as H. Masur proved, does not have negative curvature in any reasonable sense [Ber74], §3 (E). If  $g = 1$ , the Teichmüller metric coincides with the Poincaré metric on the modular surface  $R_1$  [Nag88], 2.6.5, in particular it is Riemannian with constant negative curvature.

There is a natural action of the Lie group  $GL(2, \mathbb{R})$  on  $Q_g$  (see also [HS05], §1.4 or [Mas05], §3) which is defined as follows. The map

$$q \rightarrow (\mathcal{F}_q, \mathcal{F}_{-q}), \quad q \in Q_g,$$

is a bijection between the space  $Q_g$  and the space of all pairs of transverse measured foliations. We recall that transversality for a pair  $(\mathcal{F}, \mathcal{F}^\perp)$  of measured foliations is taken in the sense that  $\mathcal{F}$  and  $\mathcal{F}^\perp$  have a common set  $\Sigma$  of (saddle) singularities, have the same index at each singularity and are transverse in the standard sense on  $M \setminus \Sigma$ . The set  $\Sigma$  of common singularities coincides with the set  $\Sigma_q$  of zeroes of the holomorphic quadratic differential  $q \equiv (\mathcal{F}, \mathcal{F}^\perp)$ . Any pair of transverse measured foliations is determined locally by a pair  $(\eta, \eta^\perp)$  of (locally defined) transverse real-valued closed 1-forms. The group  $GL(2, \mathbb{R})$  acts naturally by left multiplication on the space of (locally defined) pairs of transverse real-valued closed 1-forms, hence it acts on the space of all pairs of transverse measured foliations and on the space of  $Q_g$  of holomorphic quadratic differentials. Such an action is equivariant with respect to the action of the mapping class group, hence it passes to the quotient  $\mathcal{M}_g$ . The Teichmüller flow  $\{G_t\}_{t \in \mathbb{R}}$  is given by the action of the diagonal subgroup  $\{\text{diag}(e^{-t}, e^t)\}_{t \in \mathbb{R}}$  on  $Q_g$  (on  $\mathcal{M}_g$ ). In other terms, if we identify holomorphic quadratic differentials with pairs of transverse measured foliations as explained above, we have:

$$(4.3) \quad G_t(\mathcal{F}_q, \mathcal{F}_{-q}) := (e^{-t}\mathcal{F}_q, e^t\mathcal{F}_{-q}).$$

In geometric terms, the action of the Teichmüller flow on quadratic differentials induces a one-parameter family of deformations of the conformal structure which consist in contracting along vertical leaves (with respect to the horizontal length) and expanding along horizontal leaves (with respect to the vertical length) by reciprocal (exponential) factors. The reader can compare the definition in terms of the  $SL(2, \mathbb{R})$ -action with the analogous description of the geodesic flow on a surface of constant negative curvature (such as the modular surface). In fact, in case  $g = 1$  the above definition reduces to the standard Lie group presentation of the geodesic flow on the modular surface: the unit sub-bundle  $\mathcal{M}_1^{(1)} \subset \mathcal{M}_1$  of all holomorphic quadratic differentials of unit total area on elliptic curves can be identified with the homogeneous space  $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$  and the geodesic flow on the modular surface is then identified with the action of the diagonal subgroup of  $SL(2, \mathbb{R})$  on  $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ .

We list below, following [Vee90], [Kon97], the main structures carried by the moduli space  $\mathcal{M}_g$  of quadratic differentials:

- (1) the moduli space  $\mathcal{M}_g$  is a stratified analytic orbifold; each stratum  $\mathcal{M}_\kappa \subset \mathcal{M}_g$  (determined by the multiplicities  $\kappa = (k_1, \dots, k_\sigma)$  of the zeroes  $\{p_1, \dots, p_\sigma\}$  of quadratic differentials) is  $GL(2, \mathbb{R})$ -invariant, hence in particular  $G_t$ -invariant;
- (2) The total area function  $A : \mathcal{M}_g \rightarrow \mathbb{R}^+$ ,

$$A(q) := \int_M |q| ,$$

is  $SL(2, \mathbb{R})$ -invariant; hence the *unit bundle*  $\mathcal{M}_g^{(1)} := A^{-1}(\{1\})$  and its strata  $\mathcal{M}_\kappa^{(1)} := \mathcal{M}_\kappa \cap \mathcal{M}_g^{(1)}$  are  $SL(2, \mathbb{R})$ -invariant and, in particular,  $G_t$ -invariant.

Let  $\mathcal{M}_\kappa$  be a stratum of *orientable* quadratic differentials, that is, quadratic differentials which are squares of holomorphic 1-forms. In this case, the natural numbers  $(k_1, \dots, k_\sigma)$  are all even.

- (3) The stratum of squares  $\mathcal{M}_\kappa$  has a locally affine structure modeled on the affine space  $H^1(M, \Sigma_\kappa; \mathbb{C})$ , with  $\Sigma_\kappa := \{p_1, \dots, p_\sigma\}$ . Local charts are given by the period map  $q \rightarrow [q^{1/2}] \in H^1(M, \Sigma_\kappa; \mathbb{C})$ .
- (4) The Lebesgue measure on the euclidean space  $H^1(M, \Sigma_\kappa; \mathbb{C})$ , normalized so that the quotient torus

$$H^1(M, \Sigma_\kappa; \mathbb{C}) / H^1(M, \Sigma_\kappa; \mathbb{Z} \oplus i\mathbb{Z})$$

has volume 1, induces an absolutely continuous  $SL(2, \mathbb{R})$ -invariant measure  $\mu_\kappa$  on  $\mathcal{M}_\kappa$ . The conditional measure  $\mu_\kappa^{(1)}$  induced on the stratum  $\mathcal{M}_\kappa^{(1)}$  is  $SL(2, \mathbb{R})$ -invariant, hence  $G_t$ -invariant.

All the above structures (the stratification, the area function, the locally affine structure on the strata of squares) lift to corresponding structures at the level of the Teichmüller space of quadratic differentials, equivariant under the action of the mapping class group.

It was discovered by W. Veech [Vee90] that  $\mathcal{M}_\kappa^{(1)}$  has in general several connected components. The connected components for the strata of abelian differentials (or equivalently of orientable quadratic differentials) have been classified completely by M. Kontsevich and A. Zorich [KZ03]. A similar classification for the case of strata of non-orientable holomorphic quadratic differentials has been recently obtained by E. Laneeau in his thesis [Lan03]. Taking this phenomenon into account, the following result holds:

**Theorem 4.1.** [Mas82], [Vee86] *The total volume of the measure  $\mu_\kappa^{(1)}$  on  $\mathcal{M}_\kappa^{(1)}$  is finite and the Teichmüller geodesic flow  $\{G_t\}_{t \in \mathbb{R}}$  is ergodic on each connected component of  $\mathcal{M}_\kappa^{(1)}$ .*

We will describe below several results about the Lyapunov structure of various cocycles over the Teichmüller geodesic flow (including the tangent cocycle). We refer the reader to the recent and excellent survey [BP05] by L. Barreira and Ya. Pesin (and references therein), which covers all the relevant results on the *theory of Lyapunov exponents*, including Oseledec's multiplicative ergodic theorem and the Oseledec-Pesin reduction theorem.

The Lyapunov spectrum of the Teichmüller flow, with respect to any ergodic invariant probability measure  $\mu$  on the moduli space, has symmetries. In fact, there exists non-negative numbers  $\lambda_1^\mu = 1 \geq \lambda_2^\mu \geq \dots \geq \lambda_g^\mu$  such that the Lyapunov spectrum of the Teichmüller flow has the following form (see §5 in [Zor96], §7 in [Kon97] or §2.3 in [Zor99]):

$$\begin{aligned}
 (4.4) \quad & 2 \geq (1 + \lambda_2^\mu) \geq \dots \geq (1 + \lambda_g^\mu) \geq \overbrace{1 = \dots = 1}^{\#(\Sigma_\kappa)-1} \geq (1 - \lambda_g^\mu) \geq \\
 & \geq \dots \geq (1 - \lambda_2^\mu) \geq 0 = 0 \geq -(1 - \lambda_2^\mu) \geq \dots \geq -(1 - \lambda_g^\mu) \geq \\
 & \geq \underbrace{-1 = \dots = -1}_{\#(\Sigma_\kappa)-1} \geq -(1 + \lambda_g^\mu) \geq \dots \geq -(1 + \lambda_2^\mu) \geq -2 .
 \end{aligned}$$

In [Vee86] Veech proved that the Teichmüller flow is *non-uniformly hyperbolic*, in the sense that all of its Lyapunov exponents, except one corresponding to the flow direction, are non-zero. By formulas (4.4) Veech's theorem can be formulated as follows:

**Theorem 4.2.** [Vee86] *The inequality*

$$(4.5) \quad \lambda_2^\mu < \lambda_1^\mu = 1 .$$

*holds if  $\mu$  is the absolutely continuous  $SL(2, \mathbb{R})$ -invariant ergodic probability measure on any connected component of a stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable quadratic differentials.*

M. Kontsevich and A. Zorich have interpreted the non-negative numbers

$$(4.6) \quad \lambda_1^\mu = 1 \geq \lambda_2^\mu \geq \dots \geq \lambda_g^\mu$$

as the non-negative Lyapunov exponents of a symplectic cocycle over the Teichmüller flow, which we now describe.

The *Kontsevich-Zorich cocycle*  $\{\Phi_t\}_{t \in \mathbb{R}}$  is a cocycle over the Teichmüller flow  $\{G_t\}_{t \in \mathbb{R}}$  on the moduli space  $\mathcal{M}_g$ , defined as the projection of the trivial cocycle

$$(4.7) \quad G_t \times \text{id} : Q_g \times H^1(M, \mathbb{R}) \rightarrow Q_g \times H^1(M, \mathbb{R})$$

onto the orbifold vector bundle  $\mathcal{H}_g^1(M, \mathbb{R})$  over  $\mathcal{M}_g$  defined as

$$(4.8) \quad \mathcal{H}_g^1(M, \mathbb{R}) := (Q_g \times H^1(M, \mathbb{R})) / \Gamma_g .$$

The mapping class group  $\Gamma_g$  acts naturally on the cohomology by pull-back.

The Kontsevich-Zorich cocycle was introduced in [Kon97] as a continuous-time version of the Zorich cocycle. The Zorich cocycle was introduced earlier by A. Zorich [Zor96], [Zor97] in order to explain polynomial deviations in the homological asymptotic behavior of typical leaves of orientable measured foliations on compact surfaces, a phenomenon he had discovered in numerical experiments [Zor94]. We recall that the real homology  $H_1(M, \mathbb{R})$  and the real cohomology  $H^1(M, \mathbb{R})$  of an orientable closed surface  $M$  are (symplectically) isomorphic by Poincaré duality.

Let  $\mathcal{M}_\kappa \subset \mathcal{M}_g$  be a stratum of orientable holomorphic quadratic differentials and  $Q_\kappa \subset Q_g$  the pull-back of the stratum  $\mathcal{M}_\kappa$  to the Teichmüller space of quadratic differentials. The *relative* Kontsevich-Zorich cocycle  $\{\hat{\Phi}_t\}_{t \in \mathbb{R}}$  is defined as the projection of the trivial cocycle

$$(4.9) \quad G_t \times \text{id} : \bigcup_{q \in Q_\kappa} \{q\} \times H^1(M, \Sigma_q; \mathbb{R}) \rightarrow \bigcup_{q \in Q_\kappa} \{q\} \times H^1(M, \Sigma_q; \mathbb{R})$$

on the orbifold vector bundle  $\mathcal{H}_\kappa^1(M, \Sigma_\kappa; \mathbb{R})$  over  $\mathcal{M}_\kappa$  defined as

$$(4.10) \quad \mathcal{H}_\kappa^1(M, \Sigma_\kappa; \mathbb{R}) := \left( \bigcup_{q \in Q_\kappa} \{q\} \times H^1(M, \Sigma_q; \mathbb{R}) \right) / \Gamma_g.$$

By a similar construction it is possible to define a *punctured* Kontsevich-Zorich cocycle  $\{\Psi_t\}_{t \in \mathbb{R}}$  defined as the projection of the trivial cocycle

$$(4.11) \quad G_t \times \text{id} : \bigcup_{q \in Q_\kappa} \{q\} \times H^1(M \setminus \Sigma_q; \mathbb{R}) \rightarrow \bigcup_{q \in Q_\kappa} \{q\} \times H^1(M \setminus \Sigma_q; \mathbb{R})$$

onto the orbifold vector bundle  $\mathcal{H}_\kappa^1(M \setminus \Sigma_\kappa; \mathbb{R})$  over  $\mathcal{M}_\kappa$  defined as

$$(4.12) \quad \mathcal{H}_\kappa^1(M \setminus \Sigma_\kappa; \mathbb{R}) := \left( \bigcup_{q \in Q_\kappa} \{q\} \times H^1(M \setminus \Sigma_q; \mathbb{R}) \right) / \Gamma_g.$$

However, by Poincaré-Lefschetz duality, there exists a natural isomorphism

$$(4.13) \quad H^1(M, \Sigma_\kappa; \mathbb{R}) \equiv H^1(M \setminus \Sigma_\kappa; \mathbb{R})^*,$$

hence the punctured Kontsevich-Zorich cocycle  $\{\Psi_t\}_{t \in \mathbb{R}}$  is isomorphic to the dual  $\{\hat{\Phi}_t^*\}_{t \in \mathbb{R}}$  of the relative Kontsevich-Zorich cocycle.

Let  $\mathcal{H}_\kappa^1(M, \mathbb{R})$  be the restriction to the stratum  $\mathcal{M}_\kappa^{(1)}$  of the bundle  $\mathcal{H}_g^1(M, \mathbb{R})$  defined in (4.8). Let  $\mathcal{K}_\kappa^1(M, \Sigma_\kappa; \mathbb{R})$  be the bundle defined as the kernel of the natural bundle map  $\mathcal{H}_\kappa^1(M, \Sigma_\kappa; \mathbb{R}) \rightarrow \mathcal{H}_\kappa^1(M, \mathbb{R})$ .

**Lemma 4.3.** *The sub-bundle  $\mathcal{K}_\kappa^1(M, \Sigma_\kappa; \mathbb{R})$  is invariant under the relative Kontsevich-Zorich cocycle  $\{\hat{\Phi}_t\}_{t \in \mathbb{R}}$ . The Lyapunov spectrum of the restriction  $\{\hat{\Phi}_t|_{\mathcal{K}_\kappa^1(M, \Sigma_\kappa; \mathbb{R})}\}$  consists of the single exponent 0 with multiplicity  $\#(\Sigma_\kappa) - 1$ . In fact, there exists a Lyapunov norm on the bundle  $\mathcal{K}_\kappa^1(M, \Sigma_\kappa; \mathbb{R})$  for which the cocycle is isometric.*

*Proof.* By de Rham theorem for the relative cohomology, the relative cohomology complex  $H^*(M, \Sigma_q; \mathbb{R})$  is isomorphic to the cohomology of the complex of *relative* differential forms, which are defined as all differential forms vanishing at  $\Sigma_q$ . The map  $H^*(M, \Sigma_q; \mathbb{R}) \rightarrow H^*(M, \mathbb{R})$  is naturally defined on the de Rham cohomology since every closed, exact relative form is also a closed, respectively exact form on  $M$ . It follows that any class  $c \in H^1(M, \Sigma_q; \mathbb{R})$  which belongs to the kernel  $K^1(M, \Sigma_q; \mathbb{R})$  of the map  $H^1(M, \Sigma_q; \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$  can be represented by a differential 1-form exact on  $M$ . Hence for any  $c \in K^1(M, \Sigma_q; \mathbb{R})$  there exists a smooth function  $f_c : M \rightarrow \mathbb{R}$ , uniquely determined up to the addition of any function vanishing at  $\Sigma_q$ , such that  $c = [df_c]$  as a relative de Rham cohomology class.

As a consequence of the above discussion, the following formula yields a well-defined euclidean norm  $\|\cdot\|_K$  on  $K^1(M, \Sigma_q; \mathbb{R})$ :

$$(4.14) \quad \|c\|_K^2 := \sum_{p_1, p_2 \in \Sigma_q} |f_c(p_1) - f_c(p_2)|^2, \quad c = [df_c] \in K^1(M, \Sigma_q; \mathbb{R}).$$

The norm (4.14) induces an euclidean norm on the bundle  $\mathcal{K}_\kappa^1(M, \Sigma_\kappa; \mathbb{R})$  which is invariant under the relative Kontsevich-Zorich cocycle.  $\square$

Since the vector bundle  $\mathcal{H}_g^1(M, \mathbb{R})$  has a symplectic structure, given by the intersection form on its fibers, which are isomorphic to the cohomology  $H^1(M, \mathbb{R})$ , the Lyapunov spectrum of the cocycle  $\{\Phi_t\}_{t \in \mathbb{R}}$  with respect to any  $G_t$ -invariant ergodic probability measure  $\mu$  on  $\mathcal{M}_g^{(1)}$  is *symmetric*:

$$(4.15) \quad \lambda_1^\mu \geq \dots \geq \lambda_g^\mu \geq 0 \geq \lambda_{g+1}^\mu = -\lambda_g^\mu \geq \dots \geq \lambda_{2g}^\mu = -\lambda_1^\mu.$$

The non-negative Kontsevich-Zorich exponents coincide with the numbers (4.6) which appear in the Lyapunov spectrum (4.4) of the Teichmüller flow. This relation is explained for instance in [For05].

Zorich conjectured (see [Zor96]) that the exponents (4.15) are all distinct and different from zero when  $\mu$  is the canonical absolutely continuous ergodic invariant probability measure on any connected component of a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. In other terms he conjectured that the canonical measures are *KZ-hyperbolic* and *KZ-simple*, according to the following:

**Definition 4.4.** A  $G_t$ -invariant ergodic probability measure  $\mu$  on a stratum of orientable quadratic differentials will be called *KZ-hyperbolic* if the Kontsevich-Zorich cocycle  $(\{\Phi_t\}, \mu)$  is *non-uniformly hyperbolic*, in the sense that its Lyapunov exponents satisfy the inequalities

$$(4.16) \quad \lambda_1^\mu = 1 \geq \lambda_2^\mu \geq \dots \geq \lambda_g^\mu > 0.$$

A KZ-hyperbolic measure  $\mu$  on a stratum of orientable quadratic differentials will be called *KZ-simple* if the Kontsevich-Zorich cocycle  $(\{\Phi_t\}, \mu)$  is *simple*, in the sense that all inequalities (4.16) are strict.

In [For02] we have proved the following result:

**Theorem 4.5.** ([For02], Th. 8.5) *The absolutely continuous,  $SL(2, \mathbb{R})$ -invariant, ergodic probability measure on any connected component of a stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable quadratic differentials is KZ-hyperbolic.*

In [For05], §7, we have given an example of a  $SL(2, \mathbb{R})$ -invariant measure  $\mu$  on  $\mathcal{M}_3$ , supported on the closed  $SL(2, \mathbb{R})$ -orbit of a particular genus 3 branched cover of the 2-torus, such that  $\lambda_2^\mu = \lambda_3^\mu = 0$ .

A proof of the simplicity of the Zorich and Kontsevich-Zorich cocycles, which yields in particular a new independent proof of Theorem 4.5, has been recently obtained by A. Avila and M. Viana [AV05] by methods completely different from ours.

**Theorem 4.6.** [AV05] *The absolutely continuous,  $SL(2, \mathbb{R})$ -invariant, ergodic probability measure on any connected component of a stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable quadratic differentials is KZ-simple.*

The results of this paper do not depend in any way on the simplicity of the Kontsevich-Zorich cocycle, while the non-uniform hyperbolicity is crucial to the sharp estimates on the regularity of solutions of the cohomological equation proven in §5.3. However, our results can be refined by taking into account that the Kontsevich-Zorich exponents (4.16) are all distinct.

Our proof of Theorem 4.5 in [For02] yields in particular a new independent proof of a strong version of Veech's Theorem 4.2. In fact, we have proved that the strict inequality (4.5) holds for an arbitrary probability  $G_t$ -invariant measure on any stratum of orientable quadratic differentials. By combining our methods with a recent result of J. Athreya [Ath06] on large deviations of the Teichmüller flow, it is possible to prove a similar strict upper bound for the second exponent for Lebesgue almost all quadratic differentials in *any* orbit of the circle group  $SO(2, \mathbb{R})$  on any stratum  $\mathcal{M}_\kappa$  of orientable quadratic differentials.

We recall below the variational formulas for the evolution of Hodge norm of absolute cohomology classes under the Kontsevich-Zorich cocycle. Following §2 in [For02], such formulas can be written in terms of a natural  $\mathbb{R}$ -linear extension  $\hat{U}_q$  of the partial isometry  $U_q$ , defined in formula (3.6), which plays a crucial role in the construction of distributional solutions of the cohomological equation. Let  $\hat{U}_q : L_q^2(M) \rightarrow L_q^2(M)$  be the  $\mathbb{R}$ -linear

isometry defined as follows in terms of the partial isometry  $U_q$  and of the orthogonal projections  $\pi_q^\pm : L_q^2(M) \rightarrow \mathcal{M}_q^\pm$ :

$$(4.17) \quad \hat{U}_q := U_q \circ (I - \pi_q^-) = \overline{\pi_q^-}.$$

Let  $\{q_t\}_{t \in \mathbb{R}}$  be the orbit of an orientable quadratic differential  $q \in \mathcal{Q}_\kappa^{(1)}$  under the Teichmüller flow  $\{G_t\}_{t \in \mathbb{R}}$ . We remark that by the definition of the Teichmüller flow  $\{G_t\}_{t \in \mathbb{R}}$ , the area form  $\omega_t$  of the metric induced by the quadratic differential  $q_t$  is *constant* equal to  $\omega_q$  for all  $t \in \mathbb{R}$ . Hence the Hilbert space  $L_q^2(M)$  is invariant under the action of the Teichmüller flow (in fact, it is invariant under the full  $SL(2, \mathbb{R})$  action). For each  $t \in \mathbb{R}$ , let  $M_t$  the Riemann surface carrying  $q_t \in \mathcal{Q}_\kappa^{(1)}$  and let  $\mathcal{M}_t^\pm \subset L_q^2(M)$  be the space of meromorphic, respectively anti-meromorphic, functions on the Riemann surface  $M_t$ . Such spaces are respectively the kernels of the adjoints of the Cauchy-Riemann operators  $\partial_t^\mp$ , associated to the holomorphic quadratic differential  $q_t$ . The dimension of  $\mathcal{M}_t^\pm$  is constant equal to the genus  $g \geq 1$  of  $M$  (see §2.1) and it can be proved that  $\{\mathcal{M}_t^\pm\}_{t \in \mathbb{R}}$  are smooth families of  $g$ -dimensional subspaces of the fixed Hilbert space  $L_q^2(M)$ .

By (3.40) there exists a one-parameter family  $\{m_t^+\}_{t \in \mathbb{R}} \subset \mathcal{M}_t^+$  such that

$$(4.18) \quad c_t = c_{q_t}^+(m_t^+) := [\Re(m_t^+ q_t^{1/2})] \in H^1(M_t, \mathbb{R}).$$

**Lemma 4.7.** ([For02], Lemma 2.1) *The ordinary differential equation*

$$(4.19) \quad u' = \hat{U}_{q_t}(u)$$

*is well defined in  $L_q^2(M)$  and satisfies the following properties:*

- (1) *Solutions of the Cauchy problem for (4.19) exist for all times and are uniquely determined by the initial condition;*
- (2) *If  $u_t \in L_q^2(M)$  is any solution of (4.19) such that the initial condition  $u_0 \in \mathcal{M}_q^+$ , then  $u_t \in \mathcal{M}_t^+$  for all  $t \in \mathbb{R}$ .*
- (3) *Let  $m_t^+ \in \mathcal{M}_t^+$  be the unique solution of (4.19) with initial condition  $m_0^+ = m^+ \in \mathcal{M}_q^+$ . For all  $t \in \mathbb{R}$ , we have*

$$(4.20) \quad \Phi_t(c_q^+(m^+)) = c_{q_t}^+(m_t^+).$$

It follows immediately from Proposition 2.4 that, for every  $u \in L_q^2(M)$ , there exist functions  $v^\pm \in H_q^1(M)$  such that

$$(4.21) \quad u = \partial_q^+ v^+ + \pi_q^-(u) = \partial_q^- v^- + \pi_q^+(u).$$

The O. D. E. in formula (4.19) can be written explicitly, in terms of the orthogonal decompositions (4.21), as follows. Let  $\pi_t^\pm : L_q^2(M) \rightarrow \mathcal{M}_t^\pm$  denote the orthogonal projections in the (fixed) Hilbert space  $L_q^2(M)$ . By

definition, the projections  $\pi_t^\pm$  coincide with the projections  $\pi_q^\pm$  for  $q = q_t$ , for any  $t \in \mathbb{R}$ . A function  $u \in C^1(\mathbb{R}, L_q^2(M))$  satisfies equation (4.19) iff

$$(4.22) \quad \begin{cases} u_t = \partial_t^+ v_t + \pi_t^-(u_t) ; \\ \frac{d}{dt} u_t = \partial_t^- v_t - \pi_t^-(u_t) . \end{cases}$$

An immediate consequence of Lemma 4.7 is the following result on the variation of the Hodge norm of cohomology classes under the action of the Kontsevich-Zorich cocycle. Let  $B_q : L_q^2(M) \times L_q^2(M) \rightarrow \mathbb{C}$  be the complex bilinear form given by

$$(4.23) \quad B_q(u, v) := \int_M u v \omega_q , \quad \text{for all } u, v \in L_q^2(M) .$$

**Corollary 4.8.** ([For02], Lemma 2.1') *The variation of the Hodge norm  $\|c_t\|_H$ , which coincides with the  $L_q^2$ -norm  $|m_t^+|_0$  under the identification (4.18), is given by the following formulas:*

$$(4.24) \quad \begin{aligned} (a) \quad & \frac{d}{dt} |m_t^+|_0^2 = -2 \Re[B_q(m_t^+)] = -2 \Re \left[ \int_M (m_t^+)^2 \omega_q \right] ; \\ (b) \quad & \frac{d^2}{dt^2} |m_t^+|_0^2 = 4 \left\{ |\pi_t^-(m_t^+)|_0^2 - \Re \left[ \int_M (\partial_t^+ v_t) (\partial_t^- v_t) \omega_q \right] \right\} . \end{aligned}$$

The second order variational formula (4.24), (b), is crucial in our proof of *lower bounds* for the Kontsevich-Zorich exponents (see Theorem 4.5). The first order variational formula (4.24), (a), implies quite immediately an effective upper bound for the second exponent, which yields in particular the *average spectral gap* result proved in [For02], Corollary 2.2 (a generalization of Veech's Theorem 4.2 to arbitrary  $G_t$ -invariant ergodic probability measures):

**Theorem 4.9.** ([For02], Corollary 2.2) *The inequality*

$$(4.25) \quad \lambda_2^\mu < \lambda_1^\mu = 1 .$$

*holds for any  $G_t$ -invariant ergodic probability measure  $\mu$  on any connected component of any stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials.*

The above spectral gap result implies the following unique ergodicity theorem for measured foliations:

**Corollary 4.10.** *For any stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials, the set of quadratic differentials  $q \in \mathcal{M}_\kappa^{(1)}$  with minimal but not uniquely ergodic horizontal [vertical] foliation has zero measure with respect to any  $G_t$ -invariant probability measure  $\mu$  on  $\mathcal{M}_\kappa^{(1)}$ .*



In the remainder of this section we prove a *pointwise spectral gap* result which holds for almost all quadratic differentials in *any* orbit of the circle group  $SO(2, \mathbb{R})$  on any stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. The argument is based on formula (4.24), (a) and, as mentioned above, on a result of J. Athreya [Ath06] on large deviations of the Teichmüller flow.

The upper second Lyapunov exponent of the Kontsevich-Zorich cocycle at any (orientable) quadratic differentials  $q \in \mathcal{M}_\kappa^{(1)}$  is defined as follows. Let  $I_q(M, \mathbb{R}) \subset H^1(M, \mathbb{R})$  be the subspace of real dimension 2 defined as

$$(4.26) \quad I_q(M, \mathbb{R}) := \mathbb{R} \cdot \Re(q^{1/2}) + \mathbb{R} \cdot \Im(q^{1/2})$$

and let  $I_q^\perp(M, \mathbb{R})$  be the symplectic orthogonal of  $I_q(M, \mathbb{R})$  in  $H^1(M, \mathbb{R})$ , with respect to the symplectic structure induced by the intersection form:

$$(4.27) \quad I_q^\perp(M, \mathbb{R}) := \{c \in H^1(M, \mathbb{R}) \mid c \wedge [q^{1/2}] = 0\}.$$

The complementary sub-bundles  $I_\kappa(M, \mathbb{R})$  and  $I_\kappa^\perp(M, \mathbb{R}) \subset \mathcal{H}^1(M, \mathbb{R})$ , with fibers at any  $q \in \mathcal{M}_\kappa^{(1)}$  respectively equal to  $I_q(M, \mathbb{R})$  and  $I_q^\perp(M, \mathbb{R})$ , are invariant under the Kontsevich-Zorich cocycle. In fact, it is immediate to verify that the sub-bundle  $I_\kappa(M, \mathbb{R})$  is invariant under the Kontsevich-Zorich cocycle and that the Lyapunov spectrum of the restriction of the Kontsevich-Zorich cocycle to  $I_\kappa(M, \mathbb{R})$  equals  $\{1, -1\}$  (both exponents with multiplicity 1). Since the Kontsevich-Zorich cocycle is symplectic, the symplectic orthogonal bundle  $I_\kappa^\perp(M, \mathbb{R})$  is also invariant. In addition, it is not difficult to verify that 1 is the top (upper) exponent for the cocycle on the full cohomology bundle  $\mathcal{H}_\kappa^1(M, \mathbb{R})$ .

The *second upper (forward) exponent* of the Kontsevich-Zorich cocycle is the top upper (forward) Lyapunov exponent at any quadratic differential  $q \in \mathcal{M}_\kappa^{(1)}$  of the restriction of the cocycle to the sub-bundle  $I_\kappa^\perp(M, \mathbb{R})$ :

$$(4.28) \quad \lambda_2^+(q) := \limsup_{t \rightarrow +\infty} \frac{1}{t} \log \|\Phi_t|_{I_q^\perp(M, \mathbb{R})}\|_H.$$

**Theorem 4.11.** *For any stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials, there exists a measurable function  $L_\kappa : \mathcal{M}_\kappa^{(1)} \rightarrow [0, 1)$  such that for any (orientable) quadratic differential  $q \in \mathcal{M}_\kappa^{(1)}$ ,*

$$(4.29) \quad \lambda_2^+(q_\theta) \leq L_\kappa(q) < 1, \quad \text{for almost all } \theta \in S^1.$$

*Proof.* The argument follows closely the proof of Corollary 2.2 in [For02]. Under the isomorphism (3.40), the vector space  $I_q^\perp(M, \mathbb{R})$  is represented by meromorphic functions with *zero average* (orthogonal to constant functions). It can be seen that the subspace of zero average meromorphic functions is invariant under the flow of equation (4.20) or, equivalently (4.22).

By formula (a) in (4.24),

$$(4.30) \quad \frac{d}{dt} \log |m_t^+|_0^2 = -2 \frac{\Re B_q(m_t^+)}{|m_t^+|_0^2}.$$

Following [For02], we introduce a continuous function  $\Lambda_\kappa^+ : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  defined as follows: for any  $q \in \mathcal{M}_\kappa^{(1)}$ ,

$$(4.31) \quad \Lambda_\kappa^+(q) := \max \left\{ \frac{|B_q(m^+)|}{|m^+|_0^2} \mid m^+ \in \mathcal{M}_q^+ \setminus \{0\}, \int_M m^+ \omega_q = 0 \right\}.$$

Since by the Schwarz inequality,

$$(4.32) \quad |B_q(m_t^+)| = |(m_t^+, \overline{m_t^+})_q| \leq |m_t^+|_0^2,$$

the image of the function  $\Lambda_\kappa^+$  is contained in the interval  $[0, 1]$ . We claim that  $\Lambda_\kappa^+(q) < 1$  for all  $q \in \mathcal{M}_\kappa^{(1)}$ . In fact,  $\Lambda_\kappa^+(q) = 1$  if and only if there exists a *non-zero* meromorphic function with zero average  $m^+ \in \mathcal{M}_q^+$  such that  $|(m^+, \overline{m^+})_q| = |m^+|_0^2$ . A well-known property of the Schwarz inequality then implies that there exists  $u \in \mathbb{C}$  such that  $m^+ = u \overline{m^+}$ . However, it cannot be so, since  $m^+$  would be meromorphic and anti-meromorphic, hence constant, and by the zero average condition it would be zero.

It follows from formula (4.30) that, for any  $q \in \mathcal{M}_\kappa^{(1)}$ ,

$$(4.33) \quad \frac{1}{t} \log \|\Phi_t|I_q^\perp(M, \mathbb{R})\|_H \leq \frac{1}{t} \int_0^t \Lambda_\kappa^+(G_s(q)) ds.$$

Let  $q_0 \in \mathcal{M}_\kappa^{(1)}$ . By the large deviation result of J. Athreya (see [Ath06], Corollary 2.4) the following holds. For any  $\lambda < 1$  there exists a compact set  $K \subset \mathcal{M}_\kappa^{(1)}$  such that, for almost all  $q \in SO(2, \mathbb{R}) \cdot q_0$ ,

$$(4.34) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} |\{0 \leq s \leq t \mid G_s(q) \notin K\}| \leq \lambda.$$

Let  $\Lambda_K := \max\{\Lambda_\kappa(q) \mid q \in K\}$  and, for any  $(t, q) \in \mathbb{R}^+ \times \mathcal{M}_\kappa^{(1)}$ , let

$$\mathcal{E}_K(t, q) := |\{0 \leq s \leq t \mid G_s(q) \notin K\}|.$$

Since the function  $\Lambda_\kappa^+$  is continuous and  $\Lambda_\kappa^+(q) < 1$  for all  $q \in \mathcal{M}_\kappa^{(1)}$ , its maximum on any compact set is  $< 1$ , in particular  $\Lambda_K < 1$ . The following immediate inequality holds:

$$(4.35) \quad \int_0^t \Lambda_\kappa^+(G_s(q)) ds \leq (1 - \Lambda_K) \mathcal{E}_K(t, q) + t \Lambda_K.$$

It follows from (4.33), (4.34) and (4.35) that, for almost all  $q \in SO(2, \mathbb{R}) \cdot q_0$ ,

$$(4.36) \quad \limsup_{t \rightarrow +\infty} \frac{1}{t} \log \|\Phi_t|I_q^\perp(M, \mathbb{R})\|_H \leq (1 - \Lambda_K)\lambda + \Lambda_K < 1.$$

The function  $L_\kappa : \mathcal{M}_\kappa^{(1)} \rightarrow [0, 1)$  can be defined for every  $q_0 \in \mathcal{M}_\kappa^{(1)}$  as the essential supremum of the second upper Lyapunov exponent over the orbit  $SO(2, \mathbb{R}) \cdot q_0$  of the circle group. Such a function is measurable by definition and it is everywhere  $< 1$  by the above argument.  $\square$

For any Oseledec regular point  $q \in Q_\kappa^{(1)}$  of the Kontsevich-Zorich cocycle, let  $E_q^+(M, \mathbb{R})[E_q^-(M, \mathbb{R})] \subset H^1(M, \mathbb{R})$  be the unstable [stable] subspace of the Kontsevich-Zorich cocycle. Homology cycles which are Poincaré duals to cohomology classes in  $E_q^\pm(M, \mathbb{R})$  are called *Zorich cycles* for the foliation  $\mathcal{F}_{\pm q}$ . It follows from Theorem 4.5 and from the symplectic property of the cocycle that  $E_q^\pm$  are transverse Lagrangian subspaces (with respect to the intersection form), as conjectured by Zorich in [Zor96]. In [For02], Theorem 8.3 (see also [For05], Theorem 8.2) we have proved the following representation theorem:

**Theorem 4.12.** *For Lebesgue almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , we have*

$$(4.37) \quad E_q^+(M, \mathbb{R}) = H_q^{1,1}(M, \mathbb{R}), \quad E_q^-(M, \mathbb{R}) = H_{-q}^{1,1}(M, \mathbb{R}).$$

*(The Poincaré duals of Zorich cycles for a generic orientable measured foliation  $\mathcal{F}$  are represented by basic currents for  $\mathcal{F}$  of Sobolev order  $\leq 1$ ).*

We prove below a conjectural relation between Lyapunov exponents of a cohomology class under the Kontsevich-Zorich cocycle and the Sobolev regularity of the basic current representing the cohomology class. This result answers in the affirmative a question posed by the author in [For02] (Question 9.9) and, independently by M. Kontsevich (personal communication). In order to formulate our results in the greatest possible generality, we introduce the following class of measures on the moduli space.

**Definition 4.13.** A measure  $\mu$  on the moduli space  $\mathcal{M}_g$  of holomorphic quadratic differentials will be called  *$SO(2, \mathbb{R})$ -absolutely continuous* if induces absolutely continuous conditional measures on  $\pi_*(\mu)$ -almost every fiber of the fibration  $\pi : \mathcal{M}_g \rightarrow \mathcal{M}_g/SO(2, \mathbb{R})$  (with respect to the Haar/Lebesgue measure class on each fiber).

It is immediate that any  $SO(2, \mathbb{R})$ -invariant, hence *a fortiori* any  $SL(2, \mathbb{R})$ -invariant measure is  $SO(2, \mathbb{R})$ -absolutely continuous.

**Lemma 4.14.** *Let  $\mu$  be any  $SO(2, \mathbb{R})$ -absolutely continuous,  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. For  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , the unique basic current  $C \in \mathcal{B}_q^1(M)$  [ $C \in \mathcal{B}_{-q}^1(M)$ ] which represents a cohomology class  $c \in E_q^+(M, \mathbb{R})$  [ $c \in E_q^-(M, \mathbb{R})$ ] of Lyapunov exponent  $\lambda(c) > 0$  [ $\lambda(c) < 0$ ] under the Kontsevich-Zorich cocycle, has the following Sobolev regularity:*

$$C \in W_q^{-s}(M), \quad \text{for all } s > 1 - |\lambda(c)|.$$

*Proof.* The argument follows the proof of Lemma 8.2 in [For02]. The interpolation inequality (2.26) for fractional weighted Sobolev spaces allows us to estimate the Sobolev regularity of the basic current constructed there.

Let  $q \in \mathcal{M}_\kappa^{(1)}$  be any Oseledec regular point of the Kontsevich-Zorich cocycle and let  $c \in H^1(M, \mathbb{R})$  be a cohomology class. Let  $\{q_t\}_{t \in \mathbb{R}} \subset Q_\kappa^{(1)}$  denotes the lift to the Teichmüller space of the orbit  $\{G_t(q)\}_{t \in \mathbb{R}}$  of  $q$  under the Teichmüller flow. Let  $\mathcal{M}_t^+ \subset L_q^2(M)$  be the space of meromorphic functions on the Riemann surface  $M_t$  carrying the quadratic differential  $q_t \in Q_\kappa^{(1)}$ . According to the representation formula (3.40), there exists a (smooth) family  $\{m_t^+\}_{t \in \mathbb{R}} \subset \mathcal{M}_t^+$  such that, for each  $t \in \mathbb{R}$ ,

$$(4.38) \quad \Phi_t(c) = \Re[m_t^+ q_t^{1/2}].$$

By the variational formulas (4.20) and (4.22), there exists a (smooth) family  $\{v\}_{t \in \mathbb{R}} \subset H^1(M)$  of zero-average functions such that

$$(4.39) \quad \begin{cases} m_t^+ = \partial_t^+ v_t + \pi_t^-(m_t^+), \\ \frac{d}{dt} m_t^+ = \partial_t^- v_t - \pi_t^-(m_t^+). \end{cases}$$

Since  $\Phi_t(c) \equiv c \in H^1(M, \mathbb{R})$  (by the definition (4.7) of the Kontsevich-Zorich cocycle  $\{\Phi_t\}$  over the Teichmüller space), for each  $t \in \mathbb{R}$  there exists a unique zero average function  $U_t \in L_q^2(M)$  such that

$$(4.40) \quad dU_t = \Re[m_t^+ q_t^{1/2}] - \Re[m_0^+ q_0^{1/2}].$$

It follows that the family  $\{U_t\}_{t \in \mathbb{R}}$  is smooth and satisfies, by the variational formulas (4.39), the following Cauchy problem in  $L_q^2(M)$ :

$$(4.41) \quad \begin{cases} \frac{d}{dt} U_t = 2 \Re(v_t), \\ U_0 = 0, \end{cases}$$

We claim that, if  $c \in E_q^+(M, \mathbb{R})$  [ $c \in E_q^-(M, \mathbb{R})$ ] has Lyapunov exponent  $\lambda(c) > 0$  [ $\lambda(c) < 0$ ], the set  $\{U_t \mid t \leq 0\}$  [ $\{U_t \mid t \geq 0\}$ ] is a bounded subset of the Hilbert space  $H_q^s(M)$  for any  $s < |\lambda(c)|$ .

By Oseledec's theorem, for any  $0 < \lambda < |\lambda(c)|$ , there exist a measurable function  $K_\lambda > 0$  on  $\mathcal{M}_\kappa^{(1)}$  and such that

$$(4.42) \quad \|\Phi_t(c)\|_H = |m_t^+|_0 \leq K_\lambda(q) |m_0^+|_0 \exp(-\lambda|t|), \quad t \leq 0 \text{ } [t \geq 0].$$

For any (orientable) quadratic differential  $q \in Q_\kappa^{(1)}$ , let  $\|q\|$  denote the length of the shortest geodesic segment with endpoints in the set  $\Sigma_q$  (of zeroes of  $q$ ) with respect to the induced metric. By the Poincaré inequality proved in [For02], Lemma 6.9, there exists a constant  $K_{g,\sigma} > 0$  (depending on the genus  $g \geq 2$  of the Riemann surface  $M_q$  and on the cardinality  $\sigma := \#(\Sigma_q)$  of any  $q \in Q_\kappa^{(1)}$ ) such that

$$(4.43) \quad |v - \int_M v \omega_q|_0 \leq \frac{K_{g,\sigma}}{\|q\|} \mathcal{Q}(v, v), \quad \text{for all } v \in H_q^1(M).$$

By the commutativity property (2.6) of the horizontal and vertical vector fields, the Dirichlet form  $\mathcal{Q}$  of the quadratic differential can be written as

$$\mathcal{Q}(v, v) = |\partial_q^\pm v|_0^2, \quad \text{for all } v \in H_q^1(M)$$

(where  $\partial_q^\pm$  are the Cauchy-Riemann operators introduced in §2).

By the Poincaré inequality and by the orthogonality of the decompositions in (4.39) with respect to the invariant euclidean structure on  $L_q^2(M)$ , we have

$$(4.44) \quad |v_t|_0 \leq K_{g,\sigma} \|q_t\|^{-1} |\partial_t^+ v_t|_0 \leq K_{g,\sigma} \|q_t\|^{-1} |m_t^+|_0.$$

It follows by (4.41), (4.42) and (4.44) that there exists a measurable function  $K'_\lambda > 0$  on  $\mathcal{M}_\kappa^{(1)}$  such that if  $c \in E_q^+(M, \mathbb{R})$  [ $c \in E_q^-(M, \mathbb{R})$ ] the following inequality holds for all  $t \leq 0$  [ $t \geq 0$ ]:

$$(4.45) \quad \left| \frac{d}{dt} U_t \right|_0 \leq 2 |v_t|_0 \leq K'_\lambda(q) |m_0^+|_0 \|q_t\|^{-1} e^{-\lambda|t|}.$$

By (4.39) and (4.41) we also have the following straightforward estimate for the norm of the function  $\frac{d}{dt} U_t$  in the fixed Hilbert space  $H_q^1(M)$ :

$$(4.46) \quad \left| \frac{d}{dt} U_t \right|_1 \leq e^{|t|} |m_t^+|_0 \leq K_\lambda(q) |m_0^+|_0 e^{(1-\lambda)|t|}.$$

It follows from the inequalities (4.45) and (4.46), by the interpolation inequality proved in Lemma 2.10, that for any  $s \in (0, 1)$  there exists a measurable function  $K_\lambda^s > 0$  on the stratum  $Q_\kappa^{(1)}$  such that, if  $c \in E_q^+(M, \mathbb{R})$  [ $c \in E_q^-(M, \mathbb{R})$ ] the following inequality holds for all  $t \leq 0$  [ $t \geq 0$ ]:

$$(4.47) \quad \left| \frac{d}{dt} U_t \right|_s \leq e^{|t|} |m_t^+|_0 \leq K_\lambda^s(q) |m_0^+|_0 e^{(s-\lambda)|t|}.$$

Since  $U_0 = 0$ , by Minkowski's integral inequality we finally obtain the estimate:

$$(4.48) \quad |U_t|_s \leq K_\lambda^s(q) |m_0^+|_0 \int_0^{|t|} e^{(s-\lambda)\tau} \|q_\tau\|^{-1} d\tau.$$

By the *logarithmic law* for the Teichmüller geodesic flow on the moduli space, proved by H. Masur in [Mas93], the following estimate holds for almost all quadratic differentials  $q \in \mathcal{M}_\kappa^{(1)}$  (see [Mas93], Prop. 1.2):

$$(4.49) \quad \limsup_{\tau \rightarrow \pm\infty} \frac{-\log \|q_\tau\|}{\log |\tau|} \leq \frac{1}{2}.$$

As a consequence, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for any  $s < \lambda$ , the integral in formula (4.48) is uniformly bounded for  $t \leq 0$  [ $t \geq 0$ ]. Since for any  $s < |\lambda(c)|$ , there exists  $\lambda \in (s, |\lambda(c)|)$ , it follows that the family of functions  $\{U_t | t \leq 0\}$  [ $\{U_t | t \geq 0\}$ ] is uniformly bounded in the Sobolev space  $H_q^s(M)$  for any  $s < |\lambda(c)|$ , as claimed.

For any  $s < |\lambda(c)|$ , let  $U_s^+ \in H_q^s(M)$  [ $U_s^- \in H_q^s(M)$ ] be any weak limit of the family  $\{U_t\}$  as  $t \rightarrow -\infty$  [as  $t \rightarrow +\infty$ ], which exists since all bounded subsets of the separable Hilbert space  $H_q^s(M)$  are sequentially weakly compact. Since the functions  $U_t$  have zero average for all  $t \in \mathbb{R}$  and the subspace of zero average functions is closed in  $H_q^s(M)$  for all  $s > 0$ , the weak limit  $U_s^+$  [ $U_s^-$ ] has zero average. By contraction of the identity (4.40) with the horizontal vector field  $S$  [with the vertical vector field  $T$ ] we have:

$$(4.50) \quad \begin{aligned} SU_t &= -\Re(m_0^+) + e^t \Re(m_t^+), \quad t \leq 0, \\ [TU_t &= \Im(m_0^+) + e^{-t} \Re(m_t^+), \quad t \geq 0,] \end{aligned}$$

and by taking the limit as  $t \rightarrow -\infty$  [as  $t \rightarrow +\infty$ ],

$$(4.51) \quad \begin{aligned} SU_s^+ &= -\Re(m_0^+), \\ [TU_s^- &= \Im(m_0^+).] \end{aligned}$$

Since for almost all quadratic differential  $q \in Q_\kappa^{(1)}$  the horizontal foliation [the vertical foliation] is ergodic, the solution  $U^+ \in L_q^2(M)$  [ $U^- \in L_q^2(M)$ ] of the cohomological equation (4.51) is unique (if it exists). Hence there exists a unique zero-average function  $U^+ \in L_q^2(M)$  [ $U^- \in L_q^2(M)$ ], which solves the cohomological equation (4.51), such that  $U_s^+ = U^+$  [ $U_s^- = U^-$ ] for all  $s < |\lambda(c)|$ . As a consequence,  $U^+ \in H_q^s(M)$  [ $U^- \in H_q^s(M)$ ] for all  $s < |\lambda(c)|$ . The current  $C^+ \in W_q^{s-1}(M)$  [ $C^- \in W_q^{s-1}(M)$ ] uniquely determined by the identity

$$(4.52) \quad \begin{aligned} dU^+ &= C^+ - \Re[m_0^+ q^{1/2}], \\ [dU^- &= C^- - \Re[m_0^+ q^{1/2}],] \end{aligned}$$

is basic for the horizontal [vertical] foliation by Lemma 3.17 and represents the cohomology class  $c \in E_q^+(M, \mathbb{R})$  [ $c \in E_q^-(M, \mathbb{R})$ ] of Lyapunov exponent  $\lambda(c) > 0$  [ $\lambda(c) < 0$ ].  $\square$

**4.2. Distributional cocycles.** Let  $\mathcal{Q}_\kappa(M)$  be the space of all orientable quadratic differentials, holomorphic with respect to some complex structure on a closed surface  $M$ , with zeros of multiplicities  $\kappa = (k_1, \dots, k_\sigma)$ . For any  $s \in \mathbb{R}$ , there is a natural action of the group  $\text{Diff}^+(M)$  of orientation preserving diffeomorphisms on the trivial bundles

$$(4.53) \quad \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times H_q^s(M) \subset \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times \bar{H}_q^s(M).$$

In fact, any diffeomorphism  $f \in \text{Diff}^+(M)$  defines by pull-back an isomorphism  $f^* : \bar{H}_q^s(M) \rightarrow \bar{H}_{f^*(q)}^s(M)$  which maps the subspace  $H_q^s(M) \subset \bar{H}_q^s(M)$  onto  $H_{f^*(q)}^s(M) \subset \bar{H}_{f^*(q)}^s(M)$ . The quotient bundles

$$(4.54) \quad \left( \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times H_q^s(M) \right) / \text{Diff}_0^+(M),$$

$$\left( \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times \bar{H}_q^s(M) \right) / \text{Diff}_0^+(M)$$

are well-defined bundles over the stratum  $\mathcal{Q}_\kappa$  of the Teichmüller space of quadratic differentials. There is natural action of the mapping class group  $\Gamma_g$  on the bundles (4.54) induced by the action of  $\text{Diff}^+(M)$  on the bundles (4.53). The resulting quotient bundles

$$(4.55) \quad H_\kappa^s(M) := \left( \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times H_q^s(M) \right) / \text{Diff}^+(M),$$

$$\bar{H}_\kappa^s(M) := \left( \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times \bar{H}_q^s(M) \right) / \text{Diff}^+(M)$$

are well-defined bundles over the stratum  $\mathcal{M}_\kappa$  of the moduli space. We also introduce bundles of 1-currents over a stratum  $\mathcal{M}_\kappa$  as follows:

$$(4.56) \quad W_\kappa^s(M) := \left( \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times W_q^s(M) \right) / \text{Diff}^+(M),$$

$$\bar{W}_\kappa^s(M) := \left( \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times \bar{W}_q^s(M) \right) / \text{Diff}^+(M).$$

Since, for all  $q \in \mathcal{Q}_\kappa(M)$  and for all  $s \in \mathbb{R}$ , the weighted Sobolev space of 1-currents  $W_q^s(M)$  is isomorphic to the tensor product  $\mathbb{R}^2 \otimes H_q^s(M)$ ,

$$(4.57) \quad W_\kappa^s(M) \equiv \mathbb{R}^2 \otimes H_\kappa^s(M), \bar{W}_\kappa^s(M) \equiv \mathbb{R}^2 \otimes \bar{H}_\kappa^s(M)$$

Let  $\{G_t^s\}_{t \in \mathbb{R}}$  be the cocycle over the Teichmüller flow, defined as the projection onto the bundle  $H_\kappa^{-s}(M)$  of the trivial skew-product cocycle

$$(4.58) \quad G_t \times \text{id} : \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times H_q^{-s}(M) \rightarrow \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times H_q^{-s}(M) .$$

Let  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  be the cocycle over the Teichmüller flow, defined as the projection onto the bundle  $W_\kappa^{-s}(M)$  of the trivial skew-product cocycle:

$$(4.59) \quad G_t \times \text{id} : \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times W_q^{-s}(M) \rightarrow \bigcup_{q \in \mathcal{Q}_\kappa(M)} \{q\} \times W_q^{-s}(M) .$$

Such cocycles can be described as the cocycles obtained by parallel transport of distributions and 1-currents with respect to the trivial connection along the orbits of the Teichmüller flow. The following immediate identity allows to express the cocycle  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  in terms of the cocycle  $\{G_t^s\}_{t \in \mathbb{R}}$ :

$$(4.60) \quad \Phi_t^s := \text{diag}(e^{-t}, e^t) \otimes G_t^s \quad \text{on} \quad W_\kappa^{-s}(M) = \mathbb{R}^2 \otimes H_\kappa^{-s}(M) .$$

**Lemma 4.15.** *For any  $s \in \mathbb{R}$ , the spaces  $H_\kappa^s(M)$ ,  $\bar{H}_\kappa^s(M)$ ,  $W_\kappa^s(M)$  and  $\bar{W}_\kappa^s(M)$  are well-defined Hilbert bundles over a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. The flows  $\{G_t^s\}_{t \in \mathbb{R}}$  and  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  are well-defined smooth cocycles on the bundles  $H_\kappa^{-s}(M)$  and  $W_\kappa^{-s}(M)$  respectively.*

*Proof.* Let  $q_0 \in \mathcal{Q}_\kappa(M)$ . There exists a neighbourhood  $S_0 \subset M$  of the set  $\Sigma_{q_0}$  and a neighbourhood  $D_0 \subset \mathcal{Q}_\kappa(M)$  in the space of quadratic differentials with zeros of multiplicities  $\kappa = (k_1, \dots, k_\sigma)$  such that, for all  $q \in D_0$ ,  $\Sigma_q \subset S_0$  and the quadratic differential  $q$  is isotopic to  $q_0$  on  $S_0$ . Thus there exists a smooth map  $f : D_0 \rightarrow \text{Diff}^+(M)$  such that  $q = f_q^*(q_0)$  on  $S_0$ . The bundle  $H_\kappa^s(M)$  and  $\bar{H}_\kappa^s(M)$  are trivialized over  $D_0$  by the map

$$(4.61) \quad \begin{aligned} \bar{H}_\kappa^s(M)|_{D_0} &\rightarrow D_0 \times \bar{H}_{q_0}^s(M); \\ (q, \mathcal{D}) &\rightarrow (q, (f_q^{-1})^*(\mathcal{D})) , \end{aligned}$$

which maps the subspace  $H_\kappa^s(M)|_{D_0}$  onto  $D_0 \times H_{q_0}^s(M)$ . It follows that  $H_\kappa^s(M)$  and  $\bar{H}_\kappa^s(M)$  are well-defined Hilbert bundles, hence so are  $W_\kappa^s(M)$  and  $\bar{W}_\kappa^s(M)$  by formula (4.57). The dynamical system  $\{G_t^s\}_{t \in \mathbb{R}}$  coincides with the product cocycle  $\{G_t \times \text{id}\}$  on  $D_0 \times H_{q_0}^{-s}(M)$  with respect to the trivialization (4.61), hence  $\{G_t^s\}_{t \in \mathbb{R}}$  and  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  are well-defined smooth cocycles. In fact, the weighted Sobolev spaces  $H_q^s(M)$  and  $W_q^s(M)$  are invariant under the action of the Teichmüller flow on the space  $\mathcal{Q}_\kappa(M)$  (although their Hilbert structure is not).  $\square$

We point out that for any  $s > 0$  there is no natural extension of the distributional cocycles  $\{G_t^s\}_{t \in \mathbb{R}}$  and  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  respectively to the bundles  $\bar{H}_\kappa^{-s}(M)$  and  $\bar{W}_\kappa^{-s}(M)$ , since the action of the Teichmüller flow on  $\mathcal{Q}_\kappa$  does not



respect the domain of the Friedrichs Laplacian, hence the trivial cocycle  $\{G_t \times \text{id}\}$  is not defined on the bundles  $D_0 \times \bar{H}_{q_0}^{-s}(M)$ .

Let  $\mathcal{J}_{\kappa,+}^s(M) [\mathcal{J}_{\kappa,-}^s(M)] \subset H_{\kappa}^{-s}(M)$  be the finite dimensional sub-bundle of *horizontally [vertically] invariant distributions*. By definition, the fiber of the bundle  $\mathcal{J}_{\kappa,+}^s(M) [\mathcal{J}_{\kappa,-}^s(M)]$  at any  $q \in \mathcal{M}_{\kappa}^{(1)}$  coincides with the vector space  $\mathcal{J}_{+q}^s(M) [\mathcal{J}_{-q}^s(M)]$  of horizontally [vertically] invariant distributions.

**Lemma 4.16.** *For any  $s \geq 0$  and for any  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_{\kappa}^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable quadratic differentials,*

- (1) *the sub-bundles  $\mathcal{J}_{\kappa,\pm}^s(M) \subset H_{\kappa}^{-s}(M)$  are  $G_t^s$ -invariant, measurable and of finite, almost everywhere constant rank;*
- (2) *the cocycle  $\{G_t^s | \mathcal{J}_{\kappa,\pm}^s(M)\}$  satisfies the Oseledec's theorem.*

*Proof.* (1) The horizontal and vertical vector fields determined by a quadratic differential  $q \in \mathcal{Q}_{\kappa}(M)$  are rescaled under the Teichmüller geodesic flow and the space  $\mathcal{S}_q(M)$  of  $q$ -tempered currents is invariant. Hence the spaces  $\mathcal{J}_{\pm q}^s(M)$  of all horizontally, respectively vertically, invariant distributions are invariant. Since the Sobolev spaces  $H_q^{-s}(M)$  are invariant, the spaces  $\mathcal{J}_{\pm q}^s(M)$  are also invariant under the Teichmüller geodesic flow.

For any  $q \in \mathcal{Q}_{\kappa}(M)$ , the space  $\mathcal{J}_{+q}^{-s}(M) [\mathcal{J}_{-q}^{-s}(M)]$  can be characterized as the perpendicular of the closure in  $H_q^s(M)$  of the range of the Lie derivative  $\mathcal{L}_{S_q} [\mathcal{L}_{T_q}]$  as a linear operator defined on the space  $H_q^{s+1}(M)$ :

$$(4.62) \quad \begin{aligned} \mathcal{J}_q^{-s}(M) &= \overline{\{S_q u \mid u \in H_q^{s+1}(M)\}}^{\perp}; \\ [\mathcal{J}_{-q}^{-s}(M)] &= \overline{\{T_q u \mid u \in H_q^{s+1}(M)\}}^{\perp}. \end{aligned}$$

Let  $q_0 \in \mathcal{Q}_{\kappa}(M)$  and let  $f : D_0 \rightarrow \text{Diff}^+(M)$  be a map defined on a neighbourhood  $D_0 \subset \mathcal{Q}_{\kappa}(M)$  of  $q_0$  which trivializes the bundles  $H_{\kappa}^{-s}(M)$  as in formula (4.61). For any fixed  $v \in H_{q_0}^{s+1}(M)$  the maps

$$(4.63) \quad q \rightarrow (f_q^{-1})^* S_q f_q^*(v) \quad \text{and} \quad q \rightarrow (f_q^{-1})^* T_q f_q^*(v)$$

are well-defined and continuous on  $D_0$  with values in  $H_{q_0}^s(M)$ . Since the Hilbert spaces  $H_q^s(M)$  are separable, the sub-spaces

$$(f_q^{-1})^* \mathcal{J}_{\pm q}^{-s}(M) \subset H_{q_0}^{-s}(M)$$

are measurable functions of the quadratic differentials  $q \in \mathcal{Q}_{\kappa}(M)$ . In fact, the orthogonal projections  $I_{\pm q}$  on  $(f_q^{-1})^* \mathcal{J}_{\pm q}^{-s}(M)$  can be constructed as follows. Let  $B_0^{s+1}$  be an orthonormal basis for  $H_{q_0}^{s+1}(M)$ . If the horizontal [vertical] foliation of  $q \in D_0$  is minimal, the subset of  $H_q^s(M)$  defined as

$$(4.64) \quad B_q^s := \{S_q f_q^*(v) \mid v \in B_0^{s+1}\} \quad [B_{-q}^s := \{T_q f_q^*(v) \mid v \in B_0^{s+1}\}]$$

is linearly independent. Since the set of  $q \in D_0$  with non-minimal horizontal or vertical foliation has measure zero, the Gram-Schmidt orthonormalization algorithm applied to the system  $(f_q^{-1})^* B_q^s [ (f_q^{-1})^* B_{-q}^s ]$  yields an orthonormal basis  $\{u_k^+(q)\}_{k \in \mathbb{N}} [ \{u_k^-(q)\}_{k \in \mathbb{N}} ]$  in  $H_{q_0}^s(M)$  of the subspace

$$(4.65) \quad (f_q^{-1})^* \overline{\{S_q v \mid v \in H_q^{s+1}(M)\}} \quad [ (f_q^{-1})^* \overline{\{T_q v \mid v \in H_q^{s+1}(M)\}} ]$$

such that, for all  $k \in \mathbb{N}$ , the functions  $u_k^\pm : D_0 \rightarrow H_{q_0}^s(M)$  are defined  $\mu$ -almost everywhere and are continuous on their domain of definition by the continuity of the maps (4.63). Let  $\{\mathcal{D}_k^\pm(q)\}_{k \in \mathbb{N}} \subset H_{q_0}^{-s}(M)$  be the (orthonormal) system dual to the orthonormal system  $\{u_k^\pm(q)\}_{k \in \mathbb{N}} \subset H_{q_0}^s(M)$ . Then we can write

$$(4.66) \quad I_{\pm q}(\mathcal{D}) = \mathcal{D} - \sum_{k \in \mathbb{N}} \langle \mathcal{D}, \mathcal{D}_k^\pm(q) \rangle_{-s} \mathcal{D}_k^\pm(q), \quad \text{for all } \mathcal{D} \in H_{q_0}^{-s}(M).$$

It is immediate to verify that formula (4.66) yields the orthogonal projections onto the subspaces  $(f_q^{-1})^* \mathcal{J}_{\pm q}^{-s}(M)$  for any  $q \in D_0$ . Such projections can therefore be obtained as a limit of operators which depend continuously on  $q \in \mathcal{Q}_\kappa(M)$  in the complement of the subsets of quadratic differential with non-minimal horizontal [vertical] foliation. Since such a set has measure zero with respect to any  $G_t$ -invariant ergodic probability measure, the measurability of the sub-bundles of invariant distributions is proved.

The rank of the bundles  $\mathcal{J}_{\kappa, \pm}^s(M)$  is finite by Lemma 3.14 and Theorem 3.21 and it is almost everywhere constant with respect to any *ergodic*  $G_t$ -invariant probability measure by definition of ergodicity.

(2) It follows from the Definition 2.7 of the weighted Sobolev norms and from the fundamental theorem of interpolation (see [LM68], Chap. 1, §5.1) applied to the interpolation family of Hilbert spaces  $\{H_q^s(M) \mid s \in [0, 1]\}$  that, for any  $s \in \mathbb{R}$  and for any  $q \in \mathcal{Q}_\kappa(M)$ , the following estimates hold:

$$(4.67) \quad \|G_t^s : H_q^{-s}(M) \rightarrow H_{G_t(q)}^{-s}(M)\| \leq e^{|s||t|}.$$

Since the bundles  $\mathcal{J}_{\kappa, \pm}^s(M)$  are finite dimensional and measurable, the Osledec's theorem applies to the cocycle  $\{G_t^s | \mathcal{J}_{\kappa, \pm}^s(M)\}$  with respect to any ergodic  $G_t$ -invariant probability measure. □

The measurable dependence of the spaces of invariant distributions on the quadratic differentials can also holds for distributional solutions of the cohomological equation with arbitrary data. In order to formulate the result, which will be relevant later on, we introduce the following:

**Definition 4.17.** For any  $s \in \mathbb{R}$  and any  $r > 0$ , the range (in  $\bar{H}_q^{-s}(M)$ ) of the horizontal [vertical] Lie derivative operator (on  $\bar{H}_q^{-r}(M)$ ) is the subspace

$$(4.68) \quad \begin{aligned} R_q^{r,s}(M) &:= \{S_q U \in \bar{H}_q^{-s}(M) \mid U \in \bar{H}_q^{-r}(M)\} \\ [R_{-q}^{r,s}(M) &:= \{T_q U \in \bar{H}_q^{-s}(M) \mid U \in \bar{H}_q^{-r}(M)\}] \end{aligned}$$

The *Green operators*  $\mathcal{U}_{\pm q}^{r,s} : R_{\pm q}^{r,s}(M) \rightarrow H_q^{-r}(M)$  are defined as follows: for any distribution  $F \in R_q^{r,s}(M)$  [ $F \in R_{-q}^{r,s}(M)$ ]:

$$(4.69) \quad \begin{aligned} \mathcal{U}_q^{r,s}(F) &:= U, \quad \text{if } S_q U = F \text{ and } U \in \mathcal{I}_q^r(M)^\perp \subset \bar{H}_q^{-r}(M); \\ [\mathcal{U}_{-q}^{r,s}(F) &:= U, \quad \text{if } T_q U = F \text{ and } U \in \mathcal{I}_{-q}^r(M)^\perp \subset \bar{H}_q^{-r}(M)]. \end{aligned}$$

By Corollary 3.4, for any  $s \in \mathbb{R}$  there exists  $r > 0$  such that, for almost all quadratic differentials in every circle orbit in a stratum  $\mathcal{M}_\kappa^{(1)}$ , the Green operators are *bounded* operators defined on the codimension 1 subspace  $\bar{\mathcal{H}}_q^{-s}(M) \subset \bar{H}_q^{-s}(M)$  of distributions vanishing on constant functions.

For any  $s \in \mathbb{R}$ , let  $\bar{\mathcal{H}}_\kappa^s(M) \subset \bar{H}_\kappa^s(M)$  be the continuous sub-bundle of distributions vanishing on constant functions:

$$\bar{\mathcal{H}}_\kappa^s(M) := \{(q, \mathcal{D}) \in \bar{H}_\kappa^s(M) \mid \langle \mathcal{D}, 1 \rangle_s = 0\}.$$

**Lemma 4.18.** *Let  $\mu$  be any  $SO(2, \mathbb{R})$ -absolutely continuous,  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. For any  $s \in \mathbb{R}$ , there exists  $r > 0$  such that the the Green operator  $\mathcal{U}_{\pm q}^{r,s}$  yield a measurable bundle map  $\mathcal{U}_{\kappa, \pm}^{r,s} : \bar{\mathcal{H}}_\kappa^{-s}(M) \rightarrow H_\kappa^{-r}(M)$ . In particular, the operator norm  $\|\mathcal{U}_{\pm q}^{r,s}\|$  yields a well-defined measurable real-valued function of the quadratic differential  $q \in \mathcal{M}_\kappa^{(1)}$ .*

*Proof.* Let  $q_0 \in \mathcal{Q}_\kappa(M)$  and let  $f : D_0 \rightarrow \text{Diff}^+(M)$  be a map defined on a neighbourhood  $D_0 \subset \mathcal{Q}_\kappa(M)$  of  $q_0$  which trivializes the bundles  $\bar{H}_\kappa^{-s}(M)$  as in formula (4.61). The argument is similar to the one given in the proof of the measurability of the sub-bundles of invariant distributions in Lemma 4.16. Let  $B_0^{r+1} \subset H_{q_0}^\infty(M)$  be a basis for the Hilbert space  $H_{q_0}^{r+1}(M)$ . The subset of  $H_q^r(M)$  defined as

$$(4.70) \quad B_q^r := \{S_q f_q^*(v) \mid v \in B_0^{r+1}\} \quad [B_{-q}^r := \{T_q f_q^*(v) \mid v \in B_0^{r+1}\}]$$

is linearly independent if the horizontal [vertical] foliation of  $q \in D_0$  is minimal. Hence, by the continuity of the maps (4.63), the Gram-Schmidt orthonormalization applied to the system  $B_q^r$  [ $B_{-q}^r$ ] yields an orthonormal basis  $\{u_k^+(q)\}_{k \in \mathbb{N}}$  [ $\{u_k^-(q)\}_{k \in \mathbb{N}}$ ] in  $H_q^r(M)$  of the subspace

$$(4.71) \quad \overline{\{S_q v \mid v \in H_q^{r+1}(M)\}} \quad [\overline{\{T_q v \mid v \in H_q^{r+1}(M)\}}]$$

such that, for all  $k \in \mathbb{N}$ , the functions  $(f^{-1})^* \circ u_k^\pm : D_0 \rightarrow H_{q_0}^r(M)$  are defined  $\mu$ -almost everywhere and continuous on their domain of definition. In fact, there exists bases  $\{v_k^\pm\}_{k \in \mathbb{N}} \subset H_{q_0}^\infty(M)$  of the Hilbert space  $H_{q_0}^{r+1}(M)$  such that the following holds. For each  $k \in \mathbb{N}$  there exists a function  $v_k^\pm : D_0 \rightarrow \text{span}(v_1^\pm, \dots, v_k^\pm)$ , defined  $\mu$ -almost everywhere and continuous on its domain of definition, such that for  $\mu$ -almost all  $q \in D_0$  the set  $\{v_k^\pm(q)\}_{k \in \mathbb{N}}$  is a basis of the Hilbert space  $H_{q_0}^{r+1}(M)$  and

$$u_k^+(q) = S_q(f^* \circ v_k^+)(q) \quad [u_k^-(q) = T_q(f^* \circ v_k^-)(q)].$$

Let  $\{\mathcal{D}_k^\pm(q)\}_{k \in \mathbb{N}} \subset H_{q_0}^{-r}(M)$  be the system dual to the linearly independent system  $\{(f_q^{-1})^* u_k^\pm(q)\}_{k \in \mathbb{N}} \subset H_{q_0}^r(M)$ . Such systems are in general not orthonormal. Then for any  $F \in \mathcal{H}_{q_0}^{-s}(M)$ , the following formula holds:

$$(4.72) \quad (f_q^{-1})^* \circ \mathcal{U}_{\pm q}^{r,s} \circ f_q^*(F) := \sum_{k \in \mathbb{N}} \langle F, v_k^\pm(q) \rangle \mathcal{D}_k^\pm(q).$$

In fact, let  $U_k^\pm(q)$  be the series on the right hand side of formula (4.72). Since  $\langle f_q^* \mathcal{D}_k^\pm(q), u_h^\pm(q) \rangle = \delta_{kh}$  for any  $k, h \in \mathbb{N}$ , it follows that

$$(4.73) \quad \begin{aligned} \langle f_q^* U_k^+(q), S_q f_q^* v_k^+(q) \rangle &= \langle F, v_k^+(q) \rangle = \langle f_q^*(F), f_q^* v_k^+(q) \rangle \\ \langle f_q^* U_k^-(q), T_q f_q^* v_k^-(q) \rangle &= \langle F, v_k^-(q) \rangle = \langle f_q^*(F), f_q^* v_k^-(q) \rangle \end{aligned}$$

which implies that  $f_q^* U_k^+(q) [f_q^* U_k^-(q)]$  is a distributional solution of the equation  $S_q u = f_q^*(F) [T_q u = f_q^*(F)]$ . Finally, since all the distributions  $f_q^* \mathcal{D}_k^+(q) [f_q^* \mathcal{D}_k^-(q)]$  vanish on the orthogonal complement of the space  $\{S_q v \mid v \in H_q^{r+1}(M)\} [\{T_q v \mid v \in H_q^{r+1}(M)\}]$  in  $H_q^r(M)$ , it follows that  $f_q^* U_k^\pm(q)$  is orthogonal to the space  $\mathcal{T}_{\pm q}^r(M)$  of invariant distributions, hence  $\mathcal{U}_{\pm q}^{r,s} \circ f_q^*(F) = f_q^* U_k^\pm(q)$ . It is also immediate from formula (4.72) that, for any  $F \in \mathcal{H}_{q_0}^{-s}(M)$ , the distribution  $(f_q^{-1})^* \circ \mathcal{U}_{\pm q}^{r,s} \circ f_q^*(F)$  is a  $\mu$ -measurable function (defined almost everywhere) of the quadratic differential  $q \in D_0$  with values in the Hilbert space  $H_{q_0}^{-r}(M)$ . In fact, by construction the functions  $\mathcal{D}_k^\pm : D_0 \rightarrow H_{q_0}^{-r}(M)$  are defined  $\mu$ -almost everywhere and continuous on their domain of definition. □

Let  $\mathcal{Z}_\kappa^s(M) \subset W_\kappa^{-s}(M)$  be the infinite dimensional sub-bundle of *closed currents* over  $\mathcal{M}_\kappa^{(1)}$ . By definition, the fiber of the bundle  $\mathcal{Z}_\kappa^s(M)$  at any  $q \in \mathcal{M}_\kappa^{(1)}$  coincides with the vector space of *closed currents*:

$$(4.74) \quad \mathcal{Z}_q^s(M) := \mathcal{Z}_q(M) \cap W_q^{-s}(M).$$

The bundle  $\mathcal{Z}_\kappa^s(M)$  and the sub-bundle  $\mathcal{E}_\kappa^s(M) \subset \mathcal{Z}_\kappa^s(M)$  of *exact currents* are smooth,  $\Phi_t^s$ -invariant sub-bundles of the bundle  $W_q^{-s}(M)$ . The quotient cocycle, defined on the  $H^{-s}$  de Rham cohomology bundle, is isomorphic

to the Kontsevich-Zorich cocycle. The latter isomorphism is the essential motivation for the formulas (4.58) and (4.60) which define, respectively, the cocycles  $\{G_t^s\}_{t \in \mathbb{R}}$  and  $\{\Phi_t^s\}_{t \in \mathbb{R}}$ . In fact, let

$$(4.75) \quad j_\kappa : \mathcal{Z}_\kappa^s(M) \rightarrow \mathcal{H}_\kappa^1(M, \mathbb{R})$$

be the natural de Rham cohomology map onto the cohomology bundle  $\mathcal{H}_\kappa^1(M, \mathbb{R})$ , defined as the restriction to the stratum  $\mathcal{M}_\kappa^{(1)}$  of the cohomology bundle  $\mathcal{H}_g^1(M, \mathbb{R})$  introduced in formula (4.8).

Let  $\mathcal{B}_{\kappa, \pm}^s(M) \subset \mathcal{Z}_\kappa^s(M)$  be the sub-bundles with fiber at  $q \in \mathcal{M}_\kappa^{(1)}$  given by the vector spaces  $\mathcal{B}_{\pm q}^s(M)$  of  $\mathcal{F}_{\pm q}$ -basic currents (defined in (3.31)).

**Lemma 4.19.** *For any  $s \geq 0$  and for any  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable quadratic differentials,*

- (1) *the identity  $j_\kappa \circ \Phi_t^s = \Phi_t \circ j_\kappa$  holds everywhere on  $\mathcal{Z}_\kappa^s(M)$ ;*
- (2) *the sub-bundles  $\mathcal{B}_{\kappa, \pm}^s(M) \subset \mathcal{Z}_\kappa^s(M)$  are  $\Phi_t^s$ -invariant, measurable and of finite, almost everywhere constant rank;*
- (3) *the cocycle  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M)\}$  satisfies the Oseledec's theorem.*

*Proof.* (1) It is an immediate consequence of the definition of the cocycle  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  on the bundle  $\mathcal{Z}_\kappa^s(M)$  of closed currents.

(2) The horizontal and vertical measured foliations of a quadratic differential  $q \in \mathcal{Q}_\kappa(M)$  are projectively invariant under the Teichmüller geodesic flow and the space  $\mathcal{S}_q(M)$  of  $q$ -tempered currents is invariant. As a consequence, the spaces  $\mathcal{B}_{+q}(M)$  and  $\mathcal{B}_{-q}(M)$  of horizontally, respectively vertically, basic currents are invariant. The Sobolev spaces  $W_q^{-s}(M)$  are also invariant (although the Hilbert structure is not). Hence the spaces  $\mathcal{B}_{\pm q}^s(M)$  are invariant under the Teichmüller geodesic flow.

By Lemma 3.14, the measurability of the bundles  $\mathcal{B}_{\kappa, \pm}^s(M) \subset W_\kappa^{-s}(M)$  of basic currents is equivalent to the measurability of the bundles  $\mathcal{J}_{\kappa, \pm}^s(M) \subset H_q^{-s}(M)$ , proved in Lemma 4.16.

The rank of the bundles  $\mathcal{B}_{\kappa, \pm}^s(M)$  is finite by Theorem 3.21 and it is almost everywhere constant with respect to any *ergodic*  $G_t$ -invariant probability measure by definition of ergodicity.

(3) It follows immediately from the identities (4.60) and from the bound (4.67) that, for any  $s \in \mathbb{R}$  and for any  $q \in \mathcal{Q}_\kappa(M)$ , the following estimates hold:

$$(4.76) \quad \|\Phi_t^s : W_q^{-s}(M) \rightarrow W_{G_t(q)}^{-s}(M)\| \leq e^{(|s|+1)|t|}.$$

Since the bundles  $\mathcal{B}_{\kappa, \pm}^s(M)$  are finite dimensional and measurable, the Oseledec's theorem applies to the cocycle  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M)\}$  with respect to any ergodic  $G_t$ -invariant probability measure.  $\square$

Lemma 4.16 can be generalized to the bundle of quasi-invariant distributions. Let  $\mathcal{J}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa) \subset H_\kappa^{-s}(M)$  be the bundle over  $\mathcal{M}_\kappa^{(1)}$  defined as follows: its fiber at each  $q \in \mathcal{M}_\kappa^{(1)}$  is the vector space  $\mathcal{J}_{\pm q}^s(M \setminus \Sigma_q) \subset H_q^{-s}(M)$  of quasi-invariant distributions. An argument analogous to the proof of Lemma 4.16 proves the following:

**Lemma 4.20.** *For any  $s \geq 0$  and for any  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable quadratic differentials,*

- (1) *the sub-bundles  $\mathcal{J}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa) \subset H_\kappa^{-s}(M)$  are  $G_t^s$ -invariant, measurable and of finite, almost everywhere constant rank;*
- (2) *the cocycle  $\{G_t^s | \mathcal{J}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa)\}$  satisfies the Oseledec's theorem.*

Lemma 4.19 can be generalized to the bundle of currents closed on the complement of the singular set and to the sub-bundle of quasi-basic currents. Let  $\mathcal{Z}_\kappa^s(M \setminus \Sigma_\kappa) \subset W_\kappa^{-s}(M)$  be the bundle over  $\mathcal{M}_\kappa^{(1)}$  defined as follows: its fiber at each  $q \in \mathcal{M}_\kappa^{(1)}$  is the vector space  $\mathcal{Z}^s(M \setminus \Sigma_q) \subset W_q^{-s}(M)$  of closed currents on the open manifold  $M \setminus \Sigma_q$ . The bundle  $\mathcal{Z}_\kappa^s(M \setminus \Sigma_\kappa)$  and the sub-bundle  $\mathcal{E}_\kappa^s(M \setminus \Sigma_\kappa) \subset \mathcal{Z}_\kappa^s(M \setminus \Sigma_\kappa)$  of *exact* currents are smooth,  $\Phi_t^s$ -invariant sub-bundles of the bundle  $W_q^{-s}(M)$ . The quotient cocycle, defined on the  $H^{-s}$  de Rham punctured cohomology bundle, is isomorphic to the punctured Kontsevich-Zorich cocycle. Let

$$(4.77) \quad j_\kappa : \mathcal{Z}_\kappa^s(M \setminus \Sigma_\kappa) \rightarrow \mathcal{H}_\kappa^1(M \setminus \Sigma_\kappa, \mathbb{R})$$

be the natural de Rham cohomology map onto the punctured cohomology bundle  $\mathcal{H}_\kappa^1(M \setminus \Sigma_\kappa, \mathbb{R})$  introduced in formula (4.12).

Let  $\mathcal{B}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa) \subset \mathcal{Z}_\kappa^s(M \setminus \Sigma_\kappa)$  be the sub-bundles with fiber at  $q \in \mathcal{M}_\kappa^{(1)}$  given by the vector spaces  $\mathcal{B}_{\pm q}^s(M \setminus \Sigma_q)$  of quasi-basic currents for the measured foliation  $\mathcal{F}_{\pm q}$  (defined in (3.31)). An argument analogous to the proof of Lemma 4.19 proves the following:

**Lemma 4.21.** *For any  $s \geq 0$  and for any  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable quadratic differentials,*

- (1) *the identity  $j_\kappa \circ \Phi_t^s = \Psi_t \circ j_\kappa$  holds everywhere on  $\mathcal{Z}_\kappa^s(M \setminus \Sigma_\kappa)$ ;*
- (2) *the sub-bundles  $\mathcal{B}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa) \subset \mathcal{Z}_\kappa^s(M \setminus \Sigma_\kappa)$  are  $\Phi_t^s$ -invariant, measurable and of finite, almost everywhere constant rank;*
- (3) *the cocycle  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa)\}_{t \in \mathbb{R}}$  satisfies the Oseledec's theorem.*

It follows immediately from the definitions and from Lemma 3.14 that the following cocycle isomorphisms hold:

$$(4.78) \quad \begin{aligned} \Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M) &\equiv e^{\pm t} G_t^s | \mathcal{J}_{\kappa, \pm}^s(M); \\ \Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa) &\equiv e^{\pm t} G_t^s | \mathcal{J}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa). \end{aligned}$$

As a consequence of such an isomorphism parts (2) and (3) in Lemma 4.19 and Lemma 4.21 can be immediately derived from Lemma 4.16 and Lemma 4.20 respectively. In addition, the Lyapunov spectra and the Oseledec's decomposition of the cocycles  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M)\}$  [ $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa)\}$ ] and  $\{G_t^s | \mathcal{J}_{\kappa, \pm}^s(M)\}$  [ $\{G_t^s | \mathcal{J}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa)\}$ ] can be immediately derived from one another. By part (1) in Lemma 4.19 and Lemma 4.21, by Corollary 3.20 and by the structure theorem for basic currents (Theorem 3.21), informations on the Lyapunov spectrum of the cocycles  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa)\}$  and  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M)\}$  can be derived from that of the (punctured) Kontsevich-Zorich cocycle.

For  $s = 1$ , by [For02], Lemma 8.1, and by the representation Lemma 4.14, in the non-uniformly hyperbolic case it is possible to give a quite complete description of the distributional cocycle on the bundle of basic currents.

**Lemma 4.22.** *Let  $\mu$  be any  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic measure on a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials.*

- (1) *The cocycle  $\{\Phi_t^1\}_{t \in \mathbb{R}}$  has strictly positive [strictly negative] Lyapunov spectrum, with respect to the measure  $\mu$  on  $\mathcal{M}_\kappa^{(1)}$ , on the invariant sub-bundle  $\mathcal{B}_{\kappa, +}^1(M)$  [ $\mathcal{B}_{\kappa, -}^1(M)$ ].*
- (2) *The sum  $\mathcal{B}_\kappa^1(M) := \mathcal{B}_{\kappa, +}^1(M) + \mathcal{B}_{\kappa, -}^1(M)$  of bundles over  $\mathcal{M}_\kappa^{(1)}$  is direct and the restriction of the map  $j_\kappa$  to the sub-bundle  $\mathcal{B}_\kappa^1(M)$  is  $\mu$ -almost everywhere injective.*
- (3) *The cocycle  $\{\Phi_t^1 | \mathcal{B}_\kappa^1(M)\}_{t \in \mathbb{R}}$  is isomorphic to the Kontsevich-Zorich cocycle  $\{\Phi_t\}_{t \in \mathbb{R}}$  on the real cohomology bundle  $\mathcal{H}_\kappa^1(M, \mathbb{R})$ , hence it has the same Lyapunov spectrum.*

**4.3. Lyapunov exponents.** We prove below that the spaces of all horizontally and vertically (quasi)-basic currents and invariant distributions have well-defined Oseledec decompositions at almost all point of any stratum of the moduli space (with respect to any  $G_t$ -invariant ergodic measure). We establish a fundamental relation between the Lyapunov exponents and the Sobolev order of basic currents or distributions in each Oseledec subspace. We conclude with a crucial ‘spectral gap’ result for the distributional cocycle on the bundle of exact currents which is the basis for sharp estimates on the growth of ergodic averages, hence for the construction of square-integrable solutions of the cohomological equation in §5.2.

We introduce the following:

**Definition 4.23.** A non-zero current  $C^\pm \in \mathcal{B}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa)$  will be called (*Oseledec*) *simple* if it belongs to an Oseledec subspace of the cocycle  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M \setminus \Sigma_\kappa)\}$ . A non-zero invariant distribution  $\mathcal{D}^\pm \in \mathcal{J}_\kappa^s(M \setminus \Sigma_\kappa)$

will be called (*Oseledec*) *simple* if it belongs to an Oseledec subspace of the cocycle  $\{G_t^s | \mathcal{I}_\kappa^s(M \setminus \Sigma_\kappa)\}$ .

The above definition is well-posed since for any  $r \leq s$  the cocycles  $\{\Phi_t^r\} [\{G_t^r\}]$  are the restrictions of the cocycles  $\{\Phi_t^s\} [\{G_t^s\}]$ . It is also immediate to prove that the image of any Oseledec simple basic current under the isomorphism  $\mathcal{D}_{\pm q} : \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q) \rightarrow \mathcal{I}_{\pm q}^s(M \setminus \Sigma_q)$  introduced in formula (3.34) is an Oseledec simple invariant distribution.

**Lemma 4.24.** *Let  $\mu$  be any  $G_t$ -invariant probability measure on  $\mathcal{M}_\kappa^{(1)}$  and let  $\mathcal{R}_\mu \subset \mathcal{M}_\kappa^{(1)}$  be the set of all holomorphic orientable quadratic differentials which, for all  $s \geq 0$ , are Oseledec regular point for the cocycles  $\{\Phi_t^s | \mathcal{B}_{\kappa,+}^s(M \setminus \Sigma_\kappa)\}$  and  $\{\Phi_t^s | \mathcal{B}_{\kappa,-}^s(M \setminus \Sigma_\kappa)\}$  over the Teichmüller flow  $(\{G_t\}, \mu)$ . The set  $\mathcal{R}_\mu$  has full measure and there exist measurable functions  $L_\mu^\pm : \mathcal{B}_{\kappa,\pm}^s(M \setminus \Sigma_\kappa) \rightarrow \mathbb{R}$  (with almost everywhere constant range) such that, for any  $q \in \mathcal{R}_\mu$ , the number  $L_\mu^\pm(C^\pm)$  is equal to the Lyapunov exponent of the Oseledec simple current  $C^\pm \in \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q)$  of Sobolev order  $\mathcal{O}_q^W(C^\pm) \geq 0$  (see Definition 3.12), with respect to the cocycle  $\{\Phi_t^s | \mathcal{B}_{\kappa,\pm}^s(M \setminus \Sigma_\kappa)\}$  over the flow  $(\{G_t\}, \mu)$  for any  $s > \mathcal{O}_q^W(C^\pm)$ .*

*Proof.* For any  $r \leq s$ , the embeddings  $\mathcal{B}_{\kappa,\pm}^r(M \setminus \Sigma_\kappa) \subset \mathcal{B}_{\kappa,\pm}^s(M \setminus \Sigma_\kappa)$  hold and the cocycle  $\{\Phi_t^r\}_{t \in \mathbb{R}}$  on the bundle  $W_\kappa^{-r}(M)$  coincides with restriction of the cocycle  $\{\Phi_t^s\}_{t \in \mathbb{R}}$ , defined on the bundle  $W_\kappa^{-s}(M)$ . The Oseledec's theorem holds for the cocycles  $\{\Phi_t^s | \mathcal{B}_{\kappa,\pm}^s(M \setminus \Sigma_\kappa)\}_{t \in \mathbb{R}}$  on the measurable, finite dimensional sub-bundles  $\mathcal{B}_{\kappa,\pm}^s(M \setminus \Sigma_\kappa) \subset W_\kappa^{-s}(M)$  for all  $s \geq 0$ .

Let  $\mathcal{R}_\mu \subset \mathcal{M}_\kappa^{(1)}$  be the set of points which are Oseledec regular for the cocycles  $\{\Phi_t^k | \mathcal{B}_{\kappa,\pm}^k(M \setminus \Sigma_\kappa)\}_{t \in \mathbb{R}}$  for all  $k \in \mathbb{N}$ . By the Oseledec theorem the set  $\mathcal{R}_\mu$  has full measure and, for any  $q \in \mathcal{R}_\mu$  and for all  $s \geq r$ , the Lyapunov exponents of any Oseledec simple current  $C^\pm \in \mathcal{B}_{\kappa,\pm}^r(M \setminus \Sigma_\kappa)$  with respect to the cocycles  $\{\Phi_t^r\}_{t \in \mathbb{R}}$  and  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  are well-defined and coincide. The common value  $L_\mu^\pm(C^\pm)$  of all Lyapunov exponents with respect to the cocycles  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  for  $s > \mathcal{O}_q^W(C^\pm)$  is therefore well-defined.  $\square$

By the isomorphisms (4.78) an analogous statement holds for the cocycles on the bundles of invariant distributions:

**Lemma 4.25.** *Let  $\mu$  be a  $G_t$ -invariant probability measure on  $\mathcal{M}_\kappa^{(1)}$ . There exist measurable functions  $l_\mu^\pm : \mathcal{I}_{\kappa,\pm}(M \setminus \Sigma_\kappa) \rightarrow \mathbb{R}$  (with almost everywhere constant range) such that, for any quadratic differential  $q \in \mathcal{R}_\mu$ , the number  $l_\mu^\pm(\mathcal{D}^\pm)$  is equal to the Lyapunov exponent of the Oseledec simple invariant distribution  $\mathcal{D}^\pm \in \mathcal{I}_{\pm q}(M \setminus \Sigma_q)$  of Sobolev order  $\mathcal{O}_q^H(\mathcal{D}^\pm) \geq 0$  (see Definition 2.8), with respect to the cocycle  $\{G_t^s | \mathcal{I}_{\kappa,\pm}^s(M \setminus \Sigma_\kappa)\}$  over the flow  $(\{G_t\}, \mu)$  for any  $s > \mathcal{O}_q^H(\mathcal{D}^\pm)$ . Let  $\mathcal{D}_{\pm q} : \mathcal{B}_{\pm q}(M \setminus \Sigma_q) \rightarrow \mathcal{I}_{\pm q}(M \setminus \Sigma_q)$*



be the isomorphism introduced in formula (3.34), the following identities hold:

$$(4.79) \quad l_\mu^\pm \circ \mathcal{D}_{\pm q} = L_\mu^\pm \mp 1, \quad \text{on } \mathcal{B}_{\pm q}(M \setminus \Sigma_q).$$

Lyapunov exponents of basic currents impose restrictions on their Sobolev regularity. In fact, we have:

**Lemma 4.26.** *The following inequalities hold for any quadratic differential  $q \in \mathcal{R}_\mu \subset \mathcal{M}_\kappa^{(1)}$ . For any Oseledec simple basic current  $C^\pm \in \mathcal{B}_{\pm q}(M \setminus \Sigma_q)$  and any Oseledec simple invariant distribution  $\mathcal{D}^\pm \in \mathcal{I}_{\pm q}(M \setminus \Sigma_q)$  the Sobolev order functions satisfy the following lower bounds:*

$$(4.80) \quad \mathcal{O}_q^W(C^\pm) \geq |L_\mu^\pm(C^\pm) \mp 1| \quad \text{and} \quad \mathcal{O}_q^H(\mathcal{D}^\pm) \geq |l_\mu^\pm(\mathcal{D}^\pm)|.$$

*Proof.* Since  $C^\pm \in \mathcal{B}_{\pm q}(M \setminus \Sigma_q)$ , by the identities (4.60) and by the bound (4.67) the following inequalities hold for any  $s \geq r$ :

$$(4.81) \quad |\Phi_t^s(C^\pm)|_{-s} \leq |\Phi_t^r(C^\pm)|_{-r} \leq e^{r|t|} |C^\pm|_{-r}, \quad \text{for all } t \in \mathbb{R}.$$

On the other hand, by the Oseledec's theorem, for any  $q \in \mathcal{R}_\mu$  and any  $\epsilon > 0$  there exists a constant  $K_\epsilon(q) > 0$  such that

$$(4.82) \quad \begin{aligned} |\Phi_t^s(C^\pm)|_{-s} &\geq K_\epsilon(q) e^{(L_\mu^\pm(C^\pm) - \epsilon)t} |C^\pm|_{-s}, \quad \text{for } t \geq 0; \\ |\Phi_t^s(C^\pm)|_{-s} &\geq K_\epsilon(q) e^{(L_\mu^\pm(C^\pm) + \epsilon)t} |C^\pm|_{-s}, \quad \text{for } t < 0; \end{aligned}$$

A comparison of the inequalities (4.81) and (4.82) proves the statement for the case of basic currents. The statement for invariant distributions follows immediately by Lemma 4.25 since the following identity holds:

$$\mathcal{O}_q^H \circ \mathcal{D}_{\pm q} = \mathcal{O}_q^W \quad \text{on } \mathcal{B}_{\pm q}(M \setminus \Sigma_q).$$

□

A similar argument, based on the spectral gap for almost all quadratic differentials in every circle orbit (see Theorem 4.11), proves the following:

**Theorem 4.27.** *For any quadratic differential  $q \in \mathcal{M}_\kappa^{(1)}$  there exists a real number  $s(q) > 0$  such that, for almost all  $\theta \in S^1$  and for all  $s < s(q)$*

$$(4.83) \quad \begin{aligned} \dim \mathcal{B}_{q_\theta}^s(M \setminus \Sigma_q) &= \dim \mathcal{B}_{-q_\theta}^s(M \setminus \Sigma_q) = 1; \\ \dim \mathcal{I}_{q_\theta}^s(M \setminus \Sigma_q) &= \dim \mathcal{I}_{-q_\theta}^s(M \setminus \Sigma_q) = 1. \end{aligned}$$

*Proof.* If there exists  $s \in (0, 1)$  such that (4.83) does not hold on a positive measure subset of the circle, it follows that there exists a positive measure set  $E_s \subset S^1$  such that  $\dim \mathcal{B}_{-q_\theta}^s(M \setminus \Sigma_q) > 1$ . Hence, for any  $\theta \in E_s$ , there exists a *non-vanishing* vertically (quasi)-basic current  $C_\theta \in \mathcal{B}_{-q_\theta}^s(M \setminus \Sigma_q)$  such that  $C_\theta \wedge q_\theta^{1/2} = 0$ . We claim that for almost all  $\theta \in E_s$ , the current  $C_\theta$  has non-zero cohomology class  $c_\theta \in H^1(M \setminus \Sigma_q, \mathbb{R})$ . In fact, by Theorem

3.21, if  $C_\theta$  has vanishing cohomology class, then there exists a basic current  $\hat{C}_\theta \in \mathcal{B}_{-q_\theta}^{s-1}(M \setminus \Sigma_q)$  such that  $\delta_{-q_\theta}^1(\hat{C}_\theta) = C_\theta$ . Since  $s < 1$ , if the vertical foliation  $\mathcal{F}_{q_\theta}$  is ergodic, then the current  $\hat{C}_\theta \in \mathbb{R} \cdot \mathfrak{R}(q_\theta)$  which implies  $C_\theta = 0$ , a contradiction. Since by the Keane conjecture (see [Mas82] or [Vee82]) the vertical foliation  $\mathcal{F}_{q_\theta}$  is uniquely ergodic for almost all  $\theta \in S^1$ , the claim is proved. As in formula (4.81), we have

$$(4.84) \quad |\Phi_t^s(C_\theta)|_{-s} \leq e^{s|t|-t} |C_\theta|_{-s}, \quad \text{for all } t \in \mathbb{R}.$$

Let  $\{\Psi_t\}_{t \in \mathbb{R}}$  be the punctured Kontsevich-Zorich cocycle, introduced in §4.1. By Lemma 4.19 the following identity holds:

$$[\Phi_t^s(C_\theta)] = \Psi_t(c_\theta) \in H^1(M \setminus \Sigma_q, \mathbb{R}), \quad \text{for all } t \in \mathbb{R}.$$

The de Rham punctured cohomology bundle  $\mathcal{H}_\kappa^1(M \setminus \Sigma_\kappa, \mathbb{R})$  can be endowed, for any  $s \geq 0$ , with the quotient norm  $|\cdot|_\kappa^s$  induced by the Sobolev norm on the bundle  $W_\kappa^{-s}(M)$ . It follows by the above discussion and by the estimate (4.84) that, for each  $\theta \in E_s$ , there exists a non-vanishing cohomology class  $c_\theta \in \mathcal{H}_{q_\theta}^1(M \setminus \Sigma_q, \mathbb{R})$  such that

$$(4.85) \quad \begin{aligned} c_\theta \wedge [\mathfrak{R}(q_\theta^{1/2})] &= c_\theta \wedge [\mathfrak{I}(q_\theta^{1/2})] = 0, \\ \limsup_{t \rightarrow +\infty} \frac{1}{|t|} \log |\Psi_t(c_\theta)|_\kappa^s &\leq s - 1. \end{aligned}$$

By Lemma 4.3 and by the Poincaré-Lefschetz duality (4.13) between relative and punctured cohomology, the cohomology bundle  $\mathcal{H}_\kappa^1(M, \mathbb{R})$  admits a complement in the punctured cohomology bundle  $\mathcal{H}_\kappa^1(M \setminus \Sigma_\kappa, \mathbb{R})$  on which the punctured Kontsevich-Zorich cocycle is isometric (with respect to a continuous norm). By the estimate in (4.85) on the upper Lyapunov exponent, it follows that, since  $s < 1$ , the cohomology class  $c_\theta \in \mathcal{H}_{q_\theta}^1(M, \mathbb{R})$  for all  $\theta \in E_s$ . The restriction of the punctured Kontsevich-Zorich cocycle  $\{\Psi_t\}_{t \in \mathbb{R}}$  to the bundle  $\mathcal{H}_\kappa^1(M, \mathbb{R})$  coincides with the Kontsevich-Zorich cocycle  $\{\Phi_t\}_{t \in \mathbb{R}}$ . Since the latter is symplectic, it follows from (4.85) that its upper Lyapunov exponent on the symplectic subspace  $I_{q_\theta}^\perp(M, \mathbb{R}) \subset \mathcal{H}_{q_\theta}^1(M, \mathbb{R})$  (see formulas (4.26) and (4.27)) satisfies the estimate:

$$(4.86) \quad \lambda_2^+(q_\theta) \geq 1 - s, \quad \text{for all } \theta \in E_s.$$

By Theorem 4.11 the inequality  $\lambda_2^+(q_\theta) \leq L_\kappa(q) < 1$  holds for all  $q \in \mathcal{M}_\kappa^{(1)}$  and for almost all  $\theta \in S^1$ . Since the set  $E_s$  has positive measure, it follows that  $s > 1 - L_\kappa(q) > 0$ .

□

By Lemma 4.14, Lemma 4.22 and Lemma 4.26 we derive a fundamental relation between the weighted Sobolev order and the Lyapunov exponents of Oseledec simple basic currents and invariant distributions.

**Definition 4.28.** An *Oseledec basis* for the space  $\mathcal{I}_{\pm q}^s(M) [\mathcal{B}_{\pm q}^s(M)]$  of invariant distributions [of basic currents] is a basis contained in the union of all Oseledec subspaces for the cocycle  $\{G_t^s | \mathcal{I}_{\kappa, \pm}^s(M)\} [\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M)\}]$  at any Oseledec regular point  $q \in \mathcal{M}_{\kappa}^{(1)}$ . Since the cocycles  $\{\Phi_t^s | \mathcal{I}_{\kappa, \pm}^s(M)\}$  and  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M)\}$  are isomorphic via the maps  $\mathcal{D}_{\pm q} : \mathcal{B}_{\pm q}^s(M) \rightarrow \mathcal{I}_{\pm q}^s(M)$  introduced in (3.34), any basis  $\beta_{\pm q}^s \subset \mathcal{I}_{\pm q}^s(M)$  is Oseledec if and only if the basis  $\mathcal{D}_{\pm q}(\beta_{\pm q}^s) \subset \mathcal{B}_{\pm q}^s(M)$  is Oseledec as well.

**Theorem 4.29.** For any  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic measure  $\mu$  on a stratum  $\mathcal{M}_{\kappa}^{(1)}$  of orientable quadratic differentials, for  $\mu$ -almost all  $q \in \mathcal{M}_{\kappa}^{(1)}$  and for any Oseledec basis  $\{C_1^{\pm}, \dots, C_g^{\pm}\} \subset \mathcal{B}_{\pm q}^1(M)$ :

$$(4.87) \quad \mathcal{O}_q^W(C_i^{\pm}) = 1 - |L_{\mu}^{\pm}(C_i^{\pm})| = 1 - \lambda_i^{\mu}, \quad \text{for all } i \in \{1, \dots, g\}.$$

Consequently, for any Oseledec basis  $\{\mathcal{D}_1^{\pm}, \dots, \mathcal{D}_g^{\pm}\} \subset \mathcal{I}_{\pm q}^1(M)$ ,

$$(4.88) \quad \mathcal{O}_q^H(\mathcal{D}_i^{\pm}) = |l_{\mu}^{\pm}(\mathcal{D}_i^{\pm})| = 1 - \lambda_i^{\mu}, \quad \text{for all } i \in \{1, \dots, g\}.$$

The restriction of the cocycle  $\{\Phi_t^1\}_{t \in \mathbb{R}}$  to the invariant sub-bundle  $\mathcal{Z}_{\kappa}^1(M)$  is described by the following Oseledec-type theorem, which is a straightforward generalization of [For02], Theorem 8.7, based on Lemma 4.22.

**Theorem 4.30.** For any  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic measure on a stratum  $\mathcal{M}_{\kappa}^{(1)}$  of orientable quadratic differentials, there exists a measurable  $\Phi_t^1$ -invariant splitting:

$$(4.89) \quad \mathcal{Z}_{\kappa}^1(M) = \mathcal{B}_{\kappa, +}^1(M) \oplus \mathcal{B}_{\kappa, -}^1(M) \oplus \mathcal{E}_{\kappa}^1(M).$$

- (1) The restriction of the cocycle  $\{\Phi_t^1\}_{t \in \mathbb{R}}$  to the bundle  $\mathcal{B}_{\kappa}^1(M) := \mathcal{B}_{\kappa, +}^1(M) \oplus \mathcal{B}_{\kappa, -}^1(M)$  is (measurably) isomorphic to the Kontsevich-Zorich cocycle, hence it has Lyapunov spectrum (4.15). The sub-bundle  $\mathcal{B}_{\kappa, +}^1(M)$  corresponds to the strictly positive exponents and the sub-bundle  $\mathcal{B}_{\kappa, -}^1(M)$  to the strictly negative exponents.
- (2) The Lyapunov spectrum of restriction of the cocycle  $\{\Phi_t^1\}_{t \in \mathbb{R}}$  to the infinite dimensional bundle  $\mathcal{E}_{\kappa}^1(M)$  of exact currents is reduced to the single Lyapunov exponents 0 (in fact, the cocycle is isometric with respect to a suitable continuous norm on  $\mathcal{E}_{\kappa}^1(M)$ ).

Informations on the Lyapunov structure of the restrictions of the cocycles  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  to the sub-bundles  $\mathcal{B}_{\kappa, \pm}^s(M) \subset \mathcal{B}_{\kappa, \pm}^s(M \setminus \Sigma_{\kappa}) \subset W_{\kappa}^{-s}(M)$ , for any  $s \geq 1$ , can be derived from Theorem 4.30 and from the structure theorem for basic currents (see Theorem 3.21) combined with the following result:

**Lemma 4.31.** Let  $\mu$  be any  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_{\kappa}^{(1)}$  of orientable quadratic differentials. For any  $q \in \mathcal{R}_{\mu}$ , the image under the map  $\delta_{\pm q} : \mathcal{B}_{\pm q}(M \setminus \Sigma_q) \rightarrow \mathcal{B}_{\pm q}(M \setminus \Sigma_q)$  defined by

formulas (3.61) of any simple current  $C^\pm \in \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q)$  is a simple current  $\delta_{\pm q}(C^\pm) \in \mathcal{B}_{\pm q}^{s+1}(M \setminus \Sigma_q)$ . The following identities hold for the Sobolev order map and the Lyapunov exponent map on  $\mathcal{B}_{\pm q}(M \setminus \Sigma_q)$ :

$$(4.90) \quad \begin{aligned} \mathcal{O}_q^W \circ \delta_{\pm q} &= \mathcal{O}_q^W - 1, \quad \text{for all } q \in \mathcal{M}_\kappa^{(1)}; \\ L_\mu^\pm \circ \delta_{\pm q} &= L_\mu^\pm \mp 1, \quad \text{for all } q \in \mathcal{R}_\mu \subset \mathcal{M}_\kappa^{(1)}. \end{aligned}$$

*Proof.* For any  $C^\pm \in \mathcal{B}_{\pm q}(M \setminus \Sigma_q)$ , it follows immediately by the definitions of the Sobolev spaces and of the maps  $\delta_{\pm q}$  on  $\mathcal{B}_{\pm q}(M \setminus \Sigma_q)$  that  $\delta_q(C^+) \in W_q^{-s-1}(M)$  if and only if  $C^+ \wedge \Re(q^{1/2}) \in H_q^{-s}(M)$ , hence if and only if  $C^+ \in W_q^{-s}(M)$ . Similarly,  $\delta_{-q}(C^-) \in W_q^{-s-1}(M)$  if and only if  $C^- \wedge \Im(q^{1/2}) \in H_q^{-s}(M)$ , hence if and only if  $C^- \in W_q^{-s}(M)$ . The first identity in (4.90) is therefore proved. In fact, we have proved that the maps  $\delta_{\pm q}^s$  send  $\mathcal{B}_{\pm q}^s(M \setminus \Sigma_q)$  onto the subspace of cohomologically trivial currents in  $\mathcal{B}_{\pm q}^{s+1}(M \setminus \Sigma_q)$  while the space  $\mathcal{B}_{\pm q}^s(M)$  is mapped onto the subspace of cohomologically trivial currents in  $\mathcal{B}_{\pm q}^{s+1}(M)$ .

The following identity follows immediately from the definitions: for any quadratic differential  $q \in \mathcal{M}_\kappa^{(1)}$ , for each  $s \geq 0$  and all  $t \in \mathbb{R}$ , we have

$$(4.91) \quad \delta_{\pm G_t(q)}^s \circ \Phi_t^s = e^{\pm t} (\Phi_t^{s+1} \circ \delta_{\pm q}^s) \quad \text{on } \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q).$$

Since the maps  $\delta_{\pm q}^s : \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q) \rightarrow \mathcal{B}_{\pm q}^{s+1}(M \setminus \Sigma_q)$  are embeddings, it follows from the identity (4.91) and from the Oseledec's theorem that the current  $\delta_{\pm q}^s(C^\pm) \in \mathcal{B}_{\pm q}^{s+1}(M \setminus \Sigma_q)$  is simple if  $C^\pm \in \mathcal{B}_{\pm q}^s(M \setminus \Sigma_q)$  is. It also follows that the second identity in (4.90) holds.  $\square$

An analogous statement for invariant distributions can be derived. For any  $q \in \mathcal{M}_\kappa^{(1)}$ , let  $\mathcal{L}_{\pm q} : \mathcal{S}_q(M) \rightarrow \mathcal{S}_q(M)$  denote the Lie derivative operators on the space  $\mathcal{S}_q(M)$  of all  $q$ -tempered currents: for any  $C \in \mathcal{S}_q(M)$ ,

$$(4.92) \quad \mathcal{L}_q(C) := \mathcal{L}_{S_q}(C) \quad \text{and} \quad \mathcal{L}_{-q}(C) := \mathcal{L}_{T_q}(C).$$

**Lemma 4.32.** *For any  $q \in \mathcal{M}_\kappa^{(1)}$ , the operators  $\mathcal{L}_{\pm q} : \mathcal{J}_{\pm q}(M) \rightarrow \mathcal{J}_{\pm q}(M)$  are well-defined and injective. Let  $\mu$  be any  $G_t$ -invariant ergodic probability measure on  $\mathcal{M}_\kappa^{(1)}$ . The following identities hold on the spaces  $\mathcal{J}_{\pm q}(M)$ :*

$$(4.93) \quad \begin{aligned} \mathcal{O}_q^H \circ \mathcal{L}_{\pm q} &= \mathcal{O}_q^H - 1, \quad \text{for all } q \in \mathcal{M}_\kappa^{(1)}; \\ l_\mu^\pm \circ \mathcal{L}_{\pm q} &= l_\mu^\pm \mp 1, \quad \text{for all } q \in \mathcal{R}_\mu \subset \mathcal{M}_\kappa^{(1)}. \end{aligned}$$

**Corollary 4.33.** *Let  $\mu$  be any  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable holomorphic quadratic differentials. For any  $s \geq 0$  the (finite) Lyapunov spectrum of the cocycle  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M)\}$  is a finite subset of the countable set*

$$(4.94) \quad \{\pm \lambda_1^\mu\} \cup \{\pm \lambda_i^\mu \mp j \mid 1 < i < 2g, j \in \mathbb{N}\};$$

hence the Lyapunov spectrum of the cocycle  $\{G_t^s | \mathcal{I}_{\kappa, \pm}^s(M)\}$  is a finite subset of the countable set

$$(4.95) \quad \{0\} \cup \{\pm \lambda_i^\mu \mp (j+1) \mid 1 < i < 2g, j \in \mathbb{N}\}$$

(each element of the sets (4.94) and (4.95) is taken with multiplicity one).

*Proof.* Let  $\Pi_{\kappa, +}^1(M, \mathbb{R})$  [ $\Pi_{\kappa, -}^1(M, \mathbb{R})$ ] be the continuous sub-bundles of the cohomology bundle  $\mathcal{H}_\kappa^1(M, \mathbb{R})$  over  $\mathcal{M}_\kappa^{(1)}$  which fibers at any quadratic differential  $q \in \mathcal{M}_\kappa^{(1)}$  are given by the spaces  $\Pi_{+q}^1(M, \mathbb{R})$  [ $\Pi_{-q}^1(M, \mathbb{R})$ ] defined in formula (3.52). The sub-bundles  $\Pi_{\kappa, \pm}^1(M, \mathbb{R})$  are invariant under the Kontsevich-Zorich cocycle  $\{\Phi_t\}_{t \in \mathbb{R}}$  and the Lyapunov spectrum of the restriction  $\{\Phi_t | \Pi_{\kappa, \pm}^1(M, \mathbb{R})\}$  consists of the set  $\{\pm \lambda_i^\mu \mid 1 \leq i \leq 2g-1\}$ . By formulas (3.60), the image of the bundle  $\mathcal{B}_{\kappa, \pm}^s(M)$  under the cohomology map  $j_\kappa : \mathcal{Z}_\kappa^1(M) \rightarrow \mathcal{H}_\kappa^1(M, \mathbb{R})$  is a sub-bundle of  $\Pi_{\kappa, \pm}^1(M, \mathbb{R})$  which is invariant under the Kontsevich-Zorich cocycle. By Lemma 4.19 the cocycle  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M)\}$  is mapped under the cohomology map onto a restriction of the Kontsevich-Zorich cocycle. Let  $\delta_{\kappa, \pm}^s : \mathcal{B}_{\kappa, \pm}^{s-1}(M) \rightarrow \mathcal{B}_{\kappa, \pm}^s(M)$  the measurable bundle maps defined fiber-wise for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  as the maps  $\delta_{\pm q}^s : \mathcal{B}_{\pm q}^{s-1}(M) \rightarrow \mathcal{B}_{\pm q}^s(M)$ , defined in formula (3.61). The kernel of the cohomology map  $j_\kappa$  on  $\mathcal{B}_{\kappa, \pm}^s(M)$  is a  $\Phi_t^s$ -invariant sub-bundle which coincides with the range of the map  $\delta_{\kappa, \pm}^s$ . By Lemma 4.31, the Lyapunov spectrum (4.94) of  $\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M)$  can therefore be derived by induction on  $[s] \in \mathbb{N}$ . The Lyapunov spectrum of (4.95) of  $G_t^s | \mathcal{I}_{\kappa, \pm}^s(M)$  can then be derived by the isomorphism (4.78).  $\square$

By Corollary 3.20 for any  $s > 3$  the cohomology map is surjective for almost all quadratic differentials in any circle orbit. The above result can be refined as follows:

**Corollary 4.34.** *Let  $\mu$  be any  $SO(2, \mathbb{R})$ -absolutely continuous,  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable holomorphic quadratic differentials. For any  $s > 3$ , there exists an integer vector  $h^s := (h_2^s, \dots, h_{2g-1}^s) \in \mathbb{N}^{2g-2}$  such that the Lyapunov spectrum of the cocycle  $\{\Phi_t^s | \mathcal{B}_{\kappa, \pm}^s(M)\}$  is the (finite) set*

$$(4.96) \quad \{\pm \lambda_1^\mu\} \cup \{\pm \lambda_i^\mu \mp j \mid 1 < i < 2g, 0 \leq j \leq h_i^s\};$$

hence the Lyapunov spectrum of the cocycle  $\{G_t^s | \mathcal{I}_{\kappa, \pm}^s(M)\}$  is the (finite) set

$$(4.97) \quad \{0\} \cup \{\pm \lambda_i^\mu \mp (j+1) \mid 1 < i < 2g, 0 \leq j \leq h_i^s\}$$

(each element of the sets (4.96) and (4.97) is taken with multiplicity one).

The integer vector  $h^s$  depends on the Sobolev regularity of basic currents in the Oseledec's spaces related to the Lyapunov exponents  $\{\lambda_2^\mu, \dots, \lambda_{2g-1}^\mu\}$

which come from the Kontsevich-Zorich cocycle. Hence, in particular, in case  $\mu$  is the unique absolutely continuous,  $SL(2, \mathbb{R})$ -invariant ergodic measure on a connected component of a stratum, by Theorem 4.29 the following estimate for the numbers  $(h_2^s, \dots, h_g)$  holds: for all  $i \in \{2, \dots, g\}$ ,

$$h_i^s = \max\{h | 1 - \lambda_i^\mu + h < s\}, \quad \text{if } s \notin \mathbb{N} - \lambda_i^\mu.$$

The Oseledec-Pesin theory of the cocycles  $\{G_t^s | \mathcal{I}_{\kappa, \pm}^s(M)\}$  has crucial implications for the theory of the cohomological equation. In particular, it implies that (if  $\mu$  is a KZ-hyperbolic measure) for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , Oseledec bases of the spaces  $\mathcal{I}_{\pm q}^s(M)$  of invariant distributions have strong, quantitative linear independence properties.

**Theorem 4.35.** *Let  $\mu$  be any  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable holomorphic quadratic differentials. For any  $s > 0$  and for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , let  $b_{\pm q}^s = \{\mathcal{D}_1^\pm, \dots, \mathcal{D}_{J_\pm(s)}^\pm\}$  be a Oseledec basis of the space  $\mathcal{I}_{\pm q}^s(M)$  of invariant distributions and let*

$$(4.98) \quad (l_1^\pm, \dots, l_{J_\pm(s)}^\pm) := \left( l_\mu^\pm(\mathcal{D}_1^\pm), \dots, l_\mu^\pm(\mathcal{D}_{J_\pm(s)}^\pm) \right)$$

*denote the Lyapunov spectrum of the cocycle  $\{G_t^s | \mathcal{I}_{\kappa, \pm}^s(M)\}$  over  $(\{G_t\}, \mu)$ . For any  $\epsilon > 0$ , there exists a measurable function  $K_\epsilon^s : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that the following holds. For every  $\tau \in (0, 1]$ , there exist linearly independent systems of smooth functions  $\{u_1^\pm(\tau), \dots, u_{J_\pm(s)}^\pm(\tau)\} \subset H_q^\infty(M)$  such that, for all  $i, j \in \{1, \dots, J_\pm(s)\}$  and for all  $0 \leq r \leq s$ ,*

$$(4.99) \quad \mathcal{D}_i^\pm(u_j^\pm(\tau)) = \delta_{ij} \quad \text{and} \quad |u_j^\pm(\tau)|_r \leq K_\epsilon(q) \tau^{|l_j^\pm| - r - \epsilon}.$$

*Proof.* By Corollary 4.33, for any  $G_t$ -invariant ergodic probability measure  $\mu$  on  $\mathcal{M}_\kappa^{(1)}$  and any  $s > 0$ , the Lyapunov exponents of the distributional cocycle  $\{G_t^s | \mathcal{I}_{\kappa, \pm}^s(M)\}$  are all of the same sign, namely

$$(4.100) \quad \{l_1^\pm, \dots, l_{J_\pm(s)}^\pm\} \subset \mp \mathbb{R}^+ \cup \{0\}.$$

It follows from the Oseledec's theorem that, for any  $\epsilon > 0$ , there exists a strictly positive measurable function  $C_\epsilon^{(1)} : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that, for every  $i \in \{1, \dots, J_\pm(s)\}$ , for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for all  $t \geq 0$ ,

$$(4.101) \quad |G_{\mp t}^s(\mathcal{D}_i^\pm)|_{-s} \geq C_\epsilon^{(1)}(q) e^{(|l_i^\pm| - \epsilon)t}.$$

It also follows from the Oseledec's theorem that for any  $s > 0$ , for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and any Oseledec basis  $b_{\pm q}^s \subset \mathcal{I}_{\pm q}^s(M)$ , the distortion of the (Oseledec) basis  $G_t^s(b_{\pm q}^s) \subset H_{G_t(q)}^s(M)$  grows subexponentially in time.

The *distorsion* of a basis  $b_{\pm q}^s := \{\mathcal{D}_1^\pm, \dots, \mathcal{D}_J^\pm\} \subset \mathcal{J}_{\pm q}^s(M)$  is the number

$$(4.102) \quad d_q(b_{\pm q}^s) := \sup \left\{ \frac{\sum_{i=1}^{J_\pm(s)} |c_i| |\mathcal{D}_i^\pm|_{H_q^{-s}(M)}}{\left| \sum_{i=1}^{J_\pm(s)} c_i \mathcal{D}_i^\pm \right|_{H_q^{-s}(M)}} \mid c \in \mathbb{C}^{J_\pm(s)} \right\}.$$

The Oseledec's theorem implies that, if  $b_{\pm q}^s \subset \mathcal{J}_{\pm q}^s(M)$  is an Oseledec basis, then for any  $\epsilon > 0$ , there exists a measurable function  $C_\epsilon^{(2)} : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for all  $t \in \mathbb{R}$ ,

$$(4.103) \quad d_{G_t(q)}(G_t^s(b_{\pm q}^s)) \leq C_\epsilon^{(2)}(q) e^{\epsilon|t|}.$$

By the estimates (4.101) and (4.103), it follows that, for any  $\epsilon > 0$ , there exists a measurable function  $C_\epsilon^{(3)} : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that the following holds. For every  $t \geq 0$ , since  $H_q^\infty(M)$  is dense in  $H_q^s(M)$  for any  $s \in \mathbb{R}$ , there exists a system  $\{v_1^\pm(t), \dots, v_{J_\pm(s)}^\pm(t)\} \subset H_q^s(M)$  such that, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for all  $i, j \in \{1, \dots, J_\pm(s)\}$ ,

$$(4.104) \quad \mathcal{D}_i^\pm(v_j^\pm(t)) = \delta_{ij} \quad \text{and} \quad |v_j^\pm(t)|_{H_{G_{\mp t}(q)}^s(M)} \leq C_\epsilon^{(3)}(q) e^{-(|l_j^\pm| - \epsilon)t}.$$

By the bound (4.67) on the norm  $\|G_t^s : H_q^s(M) \rightarrow H_{G_t(q)}^s(M)\|$ , it follows that, for any  $0 \leq r \leq s$  and for all  $j \in \{1, \dots, J_\pm(s)\}$ ,

$$(4.105) \quad |v_j^\pm(t)|_{H_q^r(M)} \leq C_\epsilon^{(3)}(q) e^{(r - |l_j^\pm| + \epsilon)t}, \quad \text{for any } t \geq 0.$$

It follows that the system  $\{u_1^\pm(\tau), \dots, u_{J_\pm(s)}^\pm(\tau)\} \subset H_q^\infty(M)$ , defined for any  $\tau \in (0, 1]$  by the identities

$$(4.106) \quad u_j^\pm(\tau) := v_j^\pm(-\log \tau), \quad j \in \{1, \dots, J_\pm(s)\},$$

satisfies the required properties (4.99).  $\square$

The following theorem, which can be interpreted as spectral gap result for the cocycles  $\{G_t^0\}_{t \in \mathbb{R}}$  on the bundle  $H_\kappa^0(M)$  of square-integrable functions, is the main technical result of the paper.

For any  $(\sigma, l) \in \mathbb{R}^2$ , we introduce the *upper (forward) Lyapunov norm* of a distribution  $U \in H_q^{-\sigma}(M)$  at a quadratic differential  $q \in \mathcal{M}_\kappa^{(1)}$  as the non-negative extended real number

$$(4.107) \quad \mathcal{N}_q^{\sigma, l}(U) := \sup_{t \rightarrow +\infty} e^{-lt} |G_t^\sigma(U)|_{-\sigma}$$

**Theorem 4.36.** *Let  $\mu$  be any  $SO(2, \mathbb{R})$ -absolutely continuous,  $G_t$ -invariant ergodic probability measure on a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. For any  $\sigma > 0$  and any  $l < 1$ , there exist a real number*

$\epsilon := \epsilon(\sigma, l) > 0$  and a measurable function  $C_{\sigma, l} : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and all functions  $U \in L_q^2(M)$  of zero average,

$$(4.108) \quad \mathcal{N}_q^{\sigma, -\epsilon}(U) < C_{\sigma, l}(q) \{|U|_0 + \mathcal{N}_q^{1, l}(S_q U)\}.$$

*Proof.* The outline of the argument is as follows. From the results on the cohomological equation and on the Lyapunov spectrum of the cocycles  $\{G_t^r\}_{t \in \mathbb{R}}$  on the bundles of invariant distributions we will derive (exponential) estimates on the norms  $|G_t^r(U)|_{-r}$  for a sufficiently large  $r > 0$ , along a sequence of suitable visiting times. Since  $|G_t^0(U)|_0$  is constant, as a consequence of the invariance of the  $L_q^2(M)$  norm under the Teichmüller flow, the interpolation inequality for dual weighted Sobolev norms (Corollary 2.26) implies the required (exponential) estimates on  $|G_t^\sigma(U)|_{-\sigma}$  for any  $\sigma > 0$ .

Let  $q \in \mathcal{M}_\kappa^{(1)}$  and  $U \in L_q^2(M)$  be a function of zero average. Let us assume that there exist  $l_0 < 1$  and  $\mathcal{N} \in \mathbb{R}^+$  such that the Lyapunov norm  $\mathcal{N}_q^{1, l_0}(S_q U) \leq \mathcal{N}$ . By definition (4.107) of Lyapunov norms, for any  $t \geq 0$ ,

$$(4.109) \quad |G_t^1(S_q U)|_{-1} \leq \mathcal{N} e^{l_0 t}.$$

By the spectral gap Theorem 4.9 and by Corollary 4.33, there exist  $C_1 > 0$ ,  $l_1 > 0$  and a positive measure set  $\mathcal{P}_1 \subset \mathcal{M}_\kappa^{(1)}$  such that, for all  $q \in \mathcal{P}_1$  and all  $S_q$ -invariant distributions  $\mathcal{D} \in \mathcal{H}_q^{-r}(M) \cap \mathcal{J}_q^r(M)$ , the following holds:

$$(4.110) \quad |G_t^r(\mathcal{D})|_{-r} \leq C_1 e^{-l_1 t} |\mathcal{D}|_{-r}.$$

By Lemma 4.18 on the measurability of the Green operators for the cohomological equation, there exists  $r > 0$  such that the following holds. There exists a constant  $C_2 > 0$  and a set  $\mathcal{P}_2 \subset \mathcal{P}_1$  of positive measure such that

$$(4.111) \quad \|\mathcal{U}_q^{r, 1} : \mathcal{H}_q^{-1}(M) \rightarrow H_q^{-r}(M)\| \leq C_2, \quad \text{for all } q \in \mathcal{P}_2.$$

For  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , there exists a sequence  $(t_n)_{n \in \mathbb{N}}$  of visiting times of the forward orbit  $\{G_t(q) \mid t \geq 0\}$  to the positive measure set  $\mathcal{P}_2$  such that, for each  $n \in \mathbb{N}$ , the function  $t_n : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  is measurable and

$$(4.112) \quad \lim_{n \rightarrow +\infty} \frac{t_n(q)}{n} := p > \frac{\log C_1}{l_1}.$$

For each  $n \in \mathbb{N}$  and any  $r \in \mathbb{R}$ , let us introduce the notations:

$$\begin{aligned} G_n &:= G_{t_n}, & q_n &:= G_n(q), & \{S_n, T_n\} &:= \{S_{q_n}, T_{q_n}\}, \\ G_n^r &:= G_{t_n}^r, & U_n^r &:= G_n^r(U), & F_n &:= G_n^1(S_q U), \\ H_n^r(M) &:= H_{q_n}^r(M), & \mathcal{H}_n^r(M) &:= \mathcal{H}_{q_n}^r(M), & \mathcal{J}_n^r(M) &:= \mathcal{J}_{q_n}^r(M). \end{aligned}$$

By the definition of the Teichmüller flow  $\{G_t\}$ , the orthonormal frame  $\{S_n, T_n\} = \{e^{-t_n} S_q, e^{t_n} T_q\}$  for all  $n \in \mathbb{N}$ . Hence the following cohomological equation holds:

$$(4.113) \quad S_n U_n^r = e^{-t_n} F_n \in H_n^{-1}(M).$$



For each  $n \in \mathbb{N}$  and any  $r \in \mathbb{R}$  let  $|\cdot|_{r,n}$  denote the weighted Sobolev norm on the space  $H_n^r(M)$ . It follows from equation (4.109) that

$$(4.114) \quad |S_n U_n^r|_{-1,n} \leq \mathcal{N} e^{-(1-l_0)t_n}.$$

By the definition of the Green operators and by equation (4.111), there exists a solution  $G_n \in H_n^{-r}(M)$ , orthogonal to  $\mathcal{J}_n^r(M)$ , of the cohomological equation  $S_n G_n = S_n U_n^r \in H_n^{-1}(M)$  such that

$$(4.115) \quad |G_n|_{-r,n} \leq C_2 \mathcal{N} e^{-(1-l_0)t_n}.$$

It follows that there exists  $\mathcal{D}_n \in \mathcal{J}_n^r(M) \cap \mathcal{H}_n^{-r}(M)$  such that  $U_n \in H_n^0(M)$  has the following orthogonal decomposition in the Hilbert space  $H_n^{-r}(M)$ :

$$(4.116) \quad U_n^r = G_n + \mathcal{D}_n.$$

For each  $n \in \mathbb{N}$ , let  $\pi_n : H_n^{-r}(M) \rightarrow \mathcal{J}_n^r(M)$  be the orthogonal projection onto the subspace of  $S_n$ -invariant distributions and let  $\tau_n := t_{n+1} - t_n$ . Since by the definitions  $U_{n+1}^r = G_{\tau_n}^r(U_n^r)$ , for all  $n \in \mathbb{N}$ , and

$$G_{\tau_n}^r(\mathcal{J}_n^r(M) \cap \mathcal{H}_n^{-r}(M)) \subset \mathcal{J}_{n+1}^r(M) \cap \mathcal{H}_{n+1}^{-r}(M)$$

the following recursive identity holds:

$$(4.117) \quad \mathcal{D}_{n+1} = G_{\tau_n}^r(\mathcal{D}_n) + \pi_{n+1} \circ G_{\tau_n}^r(G_n).$$

Since  $|G_{\tau_n}^r(G_n)|_{-r,n+1} \leq e^{r\tau_n} |G_n|_{-r,n}$  by the bound (4.67) on the norm of  $G_t^r : H_t^{-r}(M) \rightarrow H_{G_t(q)}^{-r}(M)$ , it follows from (4.110) and (4.115) that

$$(4.118) \quad |\mathcal{D}_{n+1}|_{-r,n+1} \leq C_1 e^{-l_1\tau_n} |\mathcal{D}_n|_{-r,n} + C_2 \mathcal{N} e^{r\tau_n - (1-l_0)t_n},$$

which implies by induction a bound of the form

$$(4.119) \quad |\mathcal{D}_{n+1}|_{-r,n+1} \leq C_1^n e^{-l_1(t_{n+1}-t_0)} |\mathcal{D}_0|_{-r,0} + C_2 \mathcal{N} \sum_{j=0}^{n-1} C_1^j e^{s_{n,j}}$$

with the sequence  $\{s_{n,j} | n \in \mathbb{N}, 0 \leq j \leq n\}$  given by the identity

$$(4.120) \quad s_{n,j} := -l_1(t_{n+1} - t_{n-j+1}) + r\tau_{n-j} - (1-l_0)t_{n-j}.$$

Since  $U \in L_q^2(M)$  and  $G_n^r(U) = G_n^0(U) \in H_n^0(M) = L_q^2(M) \subset H_n^{-r}(M)$ , the following bound holds:

$$(4.121) \quad |G_n^r(U)|_{0,n} \leq |U|_0, \quad \text{for all } n \in \mathbb{N}.$$

It follows in particular from the decomposition (4.116) that

$$(4.122) \quad |\mathcal{D}_0|_{-r,0} \leq |U_0^r|_{-r,0} \leq |G_0^r(U)|_{0,0} \leq |U|_0.$$

The main step in the argument is the proof of the following claim. There exist a positive measurable function  $C_3 : \mathcal{P}_2 \rightarrow \mathbb{R}^+$  and a real number  $l_3 := l_3(l_0, l_1) > 0$  such that

$$(4.123) \quad \sum_{j=0}^{n-1} C_1^j e^{s_{n,j}} \leq C_3(q) e^{-l_3 n}, \quad \text{for all } n \in \mathbb{N}.$$

Let  $\omega_1 < \omega < \omega_2$  be positive real numbers such that

$$(4.124) \quad \begin{aligned} (a) \quad & (1 - l_0)\omega_1 - (l_1 + r)(\omega_2 - \omega_1) > 0, \\ (b) \quad & l_1\omega_2 - l_1(\omega_2 - \omega_1) > \log C_2. \end{aligned}$$

By condition (4.112) on the sequence  $(t_n)_{n \in \mathbb{N}}$  there exists a measurable function  $n_0 : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{N}$  such that

$$(4.125) \quad \omega_1 n \leq t_n(q) \leq \omega_2 n, \quad \text{for all } n \geq n_0(q).$$

It follows that, for any  $q \in \mathcal{M}_\kappa^{(1)}$ , all  $n \geq n_0(q)$  and all  $j \leq n$ ,

$$(4.126) \quad \begin{aligned} s_{n,j} \leq & [(l_1 + r)(\omega_2 - \omega_1) - (1 - l_0)\omega_1] n \\ & + [(1 - l_0)\omega_1 - r(\omega_2 - \omega_1) - l_1\omega_2] j + l_1(\omega_2 - \omega_1) + r\omega_2. \end{aligned}$$

Let  $K := C_2 e^{(1-l_0)\omega_1 - r(\omega_2 - \omega_1) - l_1\omega_2}$ . There are two cases to consider:

$$(4.127) \quad (a) \ K \leq 1 \quad (b) \ K > 1.$$

In case (a) we immediately obtain that, for all  $n \geq n_0(q)$ ,

$$(4.128) \quad \sum_{j=n_0}^{n-1} C_1^j e^{s_{n,j}} \leq n e^{l_1(\omega_2 - \omega_1) + r\omega_2} e^{-[(1-l_0)\omega_1 - (l_1+r)(\omega_2 - \omega_1)] n},$$

in case (b) we obtain instead that

$$(4.129) \quad \sum_{j=n_0}^{n-1} C_1^j e^{s_{n,j}} \leq \frac{e^{l_1(\omega_2 - \omega_1) + r\omega_2}}{K - 1} C_1^n e^{-[l_1\omega_2 - l_1(\omega_2 - \omega_1)] n}.$$

It follows that there exist constants  $A > 0$  and  $\alpha := \alpha(l_0, l_1) > 0$  such that

$$(4.130) \quad \sum_{j=n_0}^{n-1} C_1^j e^{s_{n,j}} \leq A e^{-\alpha n}, \quad \text{for all } n \geq n_0(q).$$

By condition (4.112) on the sequence  $(t_n)_{n \in \mathbb{N}}$  and by definition (4.120)

$$(4.131) \quad \frac{s_{n,j}(q)}{n} \rightarrow -(1 - l_2)\omega, \quad \text{as } n \rightarrow +\infty,$$

for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , uniformly with respect to  $j \in \{0, \dots, n_0(q) - 1\}$ . Hence there exists a measurable function  $n_1 : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{N}$  with  $n_1 \geq n_0$  and positive constants  $B > 0$  and  $\beta := \beta(l_0, l_1) > 0$  such that

$$(4.132) \quad \sum_{j=0}^{n_0-1} C_1^j e^{s_{n,j}} \leq B e^{-\beta n}, \quad \text{for all } n \geq n_1(q).$$

The claim (4.123) then follows from the estimates (4.130) and (4.132).

It then follows from the orthogonal decomposition (4.116), from the claim (4.123), proved above, together with the upper bounds (4.115), (4.119), (4.122) and the lower bound for visiting times in (4.125), that there exist a measurable function  $C_4 : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  and a constant  $l_4 := l_4(l_0, l_1) > 0$  such that

$$(4.133) \quad |G_n^r(U)|_{-r,n} \leq C_4(q) \{|U|_0 + \mathcal{N}\} e^{-l_4 n}, \quad \text{for all } n \geq n_1(q).$$

By the *interpolation inequality* for the scale of dual weighted Sobolev norms (see Corollary 2.26), from the upper bounds (4.121) and (4.133) it follows that for any  $\sigma > 0$  there exist a measurable function  $C_\sigma : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  and a constant  $l_\sigma := l(\sigma, l_0, l_1) > 0$  such that

$$(4.134) \quad |G_n^\sigma(U)|_{-\sigma,n} \leq C_\sigma(q) \{|U|_0 + \mathcal{N}\} e^{-l_\sigma n}, \quad \text{for all } n \geq n_1(q).$$

It remains to prove that the latter bound implies the statement of the theorem. For  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for all  $t \geq t_0(q)$ , there exists a unique  $n(t, q) \in \mathbb{N}$  such that

$$(4.135) \quad t_{n(t,q)}(q) \leq t < t_{n(t,q)+1}(q).$$

Let  $\sigma > 0$  be fixed and let  $\omega_1^{(\sigma)} < \omega < \omega_2^{(\sigma)}$  be positive real numbers such that  $l_\sigma - \sigma(\omega_2^{(\sigma)} - \omega_1^{(\sigma)}) > 0$ . There exists a measurable function  $n_2^{(\sigma)} : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}$  such that  $n_2^{(s)} \geq n_1$  and, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ ,

$$(4.136) \quad \omega_1^{(\sigma)} n \leq t_n(q) \leq \omega_2^{(\sigma)} n, \quad \text{for all } n \geq n_2^{(\sigma)}(q).$$

Let  $t_1^{(\sigma)} : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  be a measurable function such that  $t_1^{(\sigma)} \geq t_0$  and  $n(t, q) \geq n_2^{(\sigma)}(q)$  if  $t \geq t_1^{(\sigma)}(q)$ , for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ . It follows from (4.135) and (4.136) that, for any  $t \geq t_1^{(\sigma)}(q)$ ,

$$(4.137) \quad n(t, q) + 1 \geq \frac{t_{n(t,q)+1}(q)}{\omega_2^{(\sigma)}} > \frac{t}{\omega_2^{(\sigma)}}.$$

It follows by (4.136) and (4.137), for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and all  $t \geq t_1^{(\sigma)}(q)$ ,

$$(4.138) \quad \tau_{n(q,t)} \leq (\omega_2^{(\sigma)} - \omega_1^{(\sigma)}) (n(q, t) + 1) + \omega_1^{(\sigma)}.$$

Since, for any  $n \in \mathbb{N}$  and any  $t_n \leq t < t_{n+1}$ ,

$$(4.139) \quad \begin{aligned} |G_t^\sigma(U)|_{-\sigma} &= |G_{t-t_n}^\sigma \circ G_{t_n}(U)|_{-\sigma} \\ &\leq e^{\sigma(t-t_n)} |G_{t_n}^\sigma(U)|_{-\sigma} \leq e^{\sigma t_n} |G_{t_n}^\sigma(U)|_{-\sigma}, \end{aligned}$$

by the upper bounds (4.134), (4.137) and (4.138) the following estimate holds. Let  $\epsilon := \epsilon(\sigma, l_0, l_1)$  be the real number defined as follows:

$$(4.140) \quad \epsilon := \frac{l_\sigma - \sigma(\omega_2^{(\sigma)} - \omega_1^{(\sigma)})}{\omega_2^{(\sigma)}} > 0.$$

For  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and all  $t \geq t_1^{(\sigma)}(q)$

$$(4.141) \quad |G_t^\sigma(U)|_{-\sigma} \leq C_\sigma(q) e^{\sigma \omega_1^{(\sigma)} + l_\sigma} \{|U|_0 + \mathcal{N}\} e^{-\epsilon t}.$$

Finally, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and all  $0 \leq t \leq t_1^{(\sigma)}(q)$ ,

$$(4.142) \quad |G_t^\sigma(U)|_{-\sigma} \leq |U|_0 \leq e^{\epsilon t_1^{(\sigma)}(q)} |U|_0 e^{-\epsilon t}.$$

The desired estimate (4.108) immediately follows from the upper bounds (4.141) and (4.142). □

## 5. SMOOTH SOLUTIONS

In this final section we prove our main theorems on Sobolev regularity of smooth solutions of the cohomological equation for translation flows. The general result which we are able to prove for any translation surface (and almost all directions) is a direct consequence of the Fourier analysis construction of distributional solutions in §§3.1 and of the uniqueness result that follows from Theorem 4.27. The sharper result which we will prove for almost all translation surfaces (and almost all directions) requires a deeper analysis of ergodic averages of translation flows. In fact, the construction of square integrable (bounded) solutions is based (as in [MMY05]) on the Gottschalk-Hedlund theorem. The required bounds on the growth of ergodic integrals are derived from the Oseledec-type result (Theorem 4.30) and the spectral gap result (Theorem 4.36) for distributional cocycles, by the methods developed in [For02] in the study of the deviation of ergodic averages. The Oseledec-type theorem was in fact already proved in [For02], while the spectral gap theorem for distributional cocycles is new.

**5.1. The general case.** In this section we prove a result on existence of smooth solutions of the cohomological equation for translation flows which holds for *any* orientable quadratic differential in almost all directions. Such a result answers a question of Marmi, Moussa and Yoccoz who asked what is the best, that is, the smallest *regularity loss* within reach of the Fourier

analysis methods of [For97]. As indicated in [MMY05], the answer is essentially that the solution loses no more than  $3+\epsilon$  derivatives (for any  $\epsilon > 0$ ) with respect to the scale of weighted Sobolev spaces introduced in §2.1. We recall that the regularity loss obtained in [MMY05] (which only holds for *almost all* quadratic differentials) is essentially  $1 + \text{BV}$ .

**Theorem 5.1.** *Let  $q \in \mathcal{M}_\kappa^{(1)}$  be any quadratic differential. Let  $k \in \mathbb{N}$  be any integer such that  $k \geq 3$  and let  $s > k$  and  $r < k - 3$ . For almost all  $\theta \in S^1$  (with respect to the Lebesgue measure), there exists a constant  $C_{r,s}(\theta) > 0$  such that the following holds. If  $f \in H_q^s(M)$  is such that  $\mathcal{D}(f) = 0$  for all  $\mathcal{D} \in \mathcal{I}_{q_\theta}^s(M)$ , the cohomological equation  $S_\theta u = f$  has a solution  $u \in H_q^r(M)$  satisfying the following estimate:*

$$(5.1) \quad |u|_r \leq C_{r,s}(\theta) |f|_s .$$

*Proof.* As a first step, we prove that, under finitely many distributional conditions on  $f \in H_q^s(M)$  with  $s > 3$ , the cohomological equation  $S_\theta u = f$  has, for almost all  $\theta \in S^1$ , a solution  $u \in H_q^{-r}(M)$  for any  $r > 0$ , which satisfies an estimate such as (5.1). Since  $\Delta_q^F f \in H_q^{s-2}(M)$  with  $s-2 > 1$ , by Theorem 3.3, for any  $r > 0$  and for almost all  $\theta \in S^1$ , there exists a solution  $U \in \bar{H}_q^{-2-r}(M)$  of the cohomological equation  $S_\theta U = \Delta_q^F f$ , which vanishes on constant functions and satisfies the bound

$$(5.2) \quad \|U\|_{-2-r} \leq C_{r,s}^{(1)}(\theta) \|\Delta_q^F f\|_{s-2} \leq C_{r,s}^{(1)}(\theta) \|f\|_s .$$

Let  $u \in \bar{H}_q^{-r}(M)$  be the unique distribution vanishing on constant functions such that  $\Delta_q^F u = U$ . Since the commutation relation  $S_\theta \Delta_q v = \Delta_q S_\theta v$  holds for any  $v \in H_q^{3+r}(M)$ , the following distributional equation holds:

$$(5.3) \quad \Delta_q (S_\theta u - f) = 0 \quad \text{in } H_q^{-3-r}(M) .$$

In addition, from the estimate (5.2) it follows immediately that

$$(5.4) \quad |u|_{-r} \leq C_{r,s}^{(1)}(\theta) |f|_s .$$

Let  $N_\theta^r \subset \bar{H}_q^{-r}(M)$  be the subspace defined as follows:

$$N_\theta^r := \{u \in H_q^{-r}(M) \mid \Delta_q S_\theta u = 0 \in H_q^{-3-r}(M)\} .$$

Since the kernel  $\mathcal{K}^r(\Delta_q) \subset H_q^{-1-r}(M)$  (which is equal to the subspace perpendicular to the range of the operator  $\Delta_q : H_q^{3+r}(M) \rightarrow H_q^{1+r}(M)$ ) and the space  $\mathcal{I}_{q_\theta}^r(M)$  of  $S_\theta$ -invariant distributions are both finite dimensional, it follows that the subspace  $N_\theta^r$  is finite dimensional (for almost all  $\theta \in S^1$ ). As a consequence, the space

$$(5.5) \quad S_\theta(N_\theta^r) := \{S_\theta u \mid u \in N_\theta^r\} \subset \mathcal{K}^r(\Delta_q)$$

is finite dimensional, hence closed in  $H_q^{-1-r}(M)$ . We claim that there exists a constant  $C_{r,s}^{(2)}(\theta) > 0$  such that the following holds. For almost all  $\theta \in S^1$ , there exists a unique distribution  $\mathcal{U}_\theta(f) \in H_q^{-r}(M)$  such that

- (a)  $\mathcal{U}_\theta(f)$  is orthogonal to the subspace  $\mathcal{J}_{q_\theta}^r(M)$ ;
- (b)  $S_\theta \mathcal{U}_\theta(f) - f \in \mathcal{K}^r(\Delta_q)$  is orthogonal to the subspace  $S_\theta(N_\theta^r)$ ;
- (c)  $\mathcal{U}_\theta(f)$  satisfies the following bound:

$$(5.6) \quad |\mathcal{U}_\theta(f)|_{-r} \leq C_{r,s}^{(2)}(\theta) |f|_s .$$

In fact, let  $u \in H_q^r(M)$  be any solution of equation (5.3) which satisfies the bound (5.4). Let  $u_1 \in H_q^{-r}(M)$  be the component of  $u$  orthogonal in  $H_q^{-r}(M)$  to the subspace  $\mathcal{J}_{q_\theta}^r(M)$  of  $S_\theta$ -invariant distributions. Since  $u - u_1 \in \mathcal{J}_{q_\theta}^r(M)$ , the distribution  $u_1$  still satisfies the equation (5.3). In addition, the norm  $|u_1|_{-r} \leq |u|_{-r}$ , hence the bound (5.4) still holds. Since  $N_\theta^r$  is finite dimensional, there exists  $u_2 \in H_q^{-r}(M)$ , a solution of (5.3), such that  $S_\theta u_2 - f$  is equal to the component of  $S_\theta u_1 - f$  orthogonal to  $S_\theta(N_\theta^r)$  in  $H_q^{-1-r}(M)$ . Let  $\mathcal{U}_\theta(f)$  be the component of  $u_2$  orthogonal to  $\mathcal{J}_{q_\theta}^r(M)$  in  $H_q^{-r}(M)$ . By construction  $\mathcal{U}_\theta(f)$  satisfies the conditions (a) and (b) above and it is uniquely determined by them. Let us prove condition (c). The linear operator  $S_\theta : N_\theta^r \rightarrow S_\theta(N_\theta^r)$  is surjective and its domain is a finite dimensional subspace, hence it has a bounded right inverse and there exists  $K_r := K_r(\theta) > 0$  such that, for any  $u \in N_\theta^r$  orthogonal to  $\mathcal{J}_{q_\theta}^r(M)$ ,

$$(5.7) \quad |u|_{-r} \leq K_r |S_\theta u|_{-1-r} .$$

Since by definition  $\mathcal{U}_\theta := \mathcal{U}_\theta(f)$  and  $u_1 \in H_q^{-r}(M)$  are both orthogonal to  $\mathcal{J}_{q_\theta}^r(M)$  and  $\mathcal{U}_\theta - u_1 \in N_\theta^r$ , by the bound (5.7),

$$(5.8) \quad |\mathcal{U}_\theta|_{-r} \leq |\mathcal{U}_\theta - u_1|_{-r} + |u_1|_{-r} \leq K_r |S_\theta(\mathcal{U}_\theta - u_1)|_{-1-r} + |u_1|_{-r} .$$

Since again by definition  $S_\theta \mathcal{U}_\theta = S_\theta u_2$  and  $S_\theta u_2 - f$  is equal to the component of  $S_\theta u_1 - f$  orthogonal to  $S_\theta(N_\theta^r)$ ,

$$(5.9) \quad \begin{aligned} |S_\theta(\mathcal{U}_\theta - u_1)|_{-1-r} &\leq |S_\theta u_1 - f|_{-1-r} + |S_\theta u_2 - f|_{-1-r} \\ &\leq 2|S_\theta u_1 - f|_{-1-r} \leq 2|u_1|_{-r} + 2|f|_{-1-r} . \end{aligned}$$

It follows from the bounds (5.8), (5.9) and from the bound (5.4) for the distribution  $u_1 \in H_q^{-r}(M)$  that the required bound (5.6) holds with

$$C_{r,s}^{(2)}(\theta) := [(2K_r(\theta) + 1)C_{r,s}^{(1)}(\theta)] + 1 .$$

Since  $S_\theta(N_\theta^r) \subset \mathcal{K}^r(\Delta_q)$  is finite dimensional, there exist a finite (linearly independent) set  $\{\Phi_1, \dots, \Phi_K\}$  of bounded linear (real-valued) functionals on the Hilbert space  $H_q^{-1-r}(M)$  such that

$$\mathcal{K}^r(\Delta_q) \cap S_\theta(N_\theta^r)^\perp \cap \text{Ker}(\Phi_1) \cap \dots \cap \text{Ker}(\Phi_K) = \{0\} .$$

Let  $\{\mathcal{D}_1, \dots, \mathcal{D}_K\} \subset H_q^{-s}(M)$  be the system of distributions defined as follows: for each  $j \in \{1, \dots, K\}$ ,

$$(5.10) \quad \mathcal{D}_j(f) := \Phi_j(S_\theta \mathcal{U}_\theta(f) - f), \quad \text{for all } f \in H_q^s(M).$$

The system  $\{\mathcal{D}_1, \dots, \mathcal{D}_K\}$  has by construction the following property: if  $\mathcal{D}_j(f) = 0$  for all  $j \in \{1, \dots, K\}$ , then  $\mathcal{U}_\theta(f)$  is the required solution of the cohomological equation  $S_\theta u = f$ . In fact, under such conditions  $\mathcal{U}_\theta(f)$  is by construction a solution orthogonal to constant functions. By Theorem 4.27, there exists  $s(q) > 0$  such that for all  $0 < s < s(q)$  the space  $\mathcal{J}_{q\theta}^s(M)$  is 1-dimensional for almost all  $\theta \in S^1$ . Thus, if  $r < s(q)$ , for almost all  $\theta \in S^1$  the solution  $u \in H_q^{-r}(M)$  of the cohomological equation  $S_\theta u = f$  is unique (if it exists). It follows that, for any  $s(q) > r > 0$ , the distribution  $\mathcal{U}_\theta(f)$  is the unique solution in  $H_q^{-r}(M)$  of the cohomological equation, which implies that  $\mathcal{U}_\theta(f) \in H_q^{-r}(M)$  for any  $r > 0$ .

We claim that  $\{\mathcal{D}_1, \dots, \mathcal{D}_K\} \subset \mathcal{J}_{q\theta}^s(M)$ . In fact, by its definition in formula (5.10), the distribution  $\mathcal{D}_j \in H_q^{-s}(M)$  for all  $j \in \{1, \dots, K\}$ . In addition, if there exists  $v \in H_q^{s+1}(M)$  such that  $f = S_\theta v$ , by conditions (a) and (b) on the distribution  $\mathcal{U}_\theta(f)$  we have

$$S_\theta(\mathcal{U}_\theta(f) - v) = S_\theta \mathcal{U}_\theta(f) - f \in \mathcal{K}^r(\Delta_q) \cap S_\theta(N_\theta^r)^\perp.$$

However, by definition  $S_\theta(\mathcal{U}_\theta(f) - v) \in S_\theta(N_\theta^r)$ , hence  $S_\theta \mathcal{U}_\theta(f) - f = 0$  which implies that  $\mathcal{D}_1(f) = \dots = \mathcal{D}_K(f) = 0$ . Hence by definition all the distributions of the system  $\{\mathcal{D}_1, \dots, \mathcal{D}_K\}$  are  $S_\theta$ -invariant.

Finally we prove the statement of the theorem by induction on  $k \in \mathbb{N}$ . For  $k = 3$  the statement holds by the previous argument. Let us assume that the statement holds for  $k \geq 3$ . By the induction hypothesis, for any  $s > k+1$  and any  $r < k-2$  there exists  $C_{r,s}(\theta) > 0$  such that, for almost all  $\theta \in S^1$  and for any  $f \in H_q^s(M)$  with  $\mathcal{D}(f) = 0$  for all  $\mathcal{D} \in \mathcal{J}_{q\theta}^s(M)$ , there exist solution  $u$ ,  $u_S$  and  $u_T \in H_q^{r-1}(M)$  of the cohomological equations  $S_\theta u = f$ ,  $S_\theta u_S = Sf$  and  $S_\theta u_T = Tf$  respectively such that  $u$ ,  $u_S$  and  $u_T$  are orthogonal to constant functions and the following bound holds:

$$(5.11) \quad |u|_{r-1}^2 + |u_S|_{r-1}^2 + |u_T|_{r-1}^2 \leq C_{r,s}(\theta) (|f|_{s-1}^2 + |Tf|_{s-1}^2 + |Sf|_{s-1}^2).$$

In fact, since  $f \in H_q^s(M)$  is such that  $\mathcal{D}(f) = 0$  for all  $\mathcal{D} \in \mathcal{J}_{q\theta}^s(M)$  it follows immediately that  $\mathcal{D}(Sf) = \mathcal{D}(Tf) = 0$  for all  $\mathcal{D} \in \mathcal{J}_{q\theta}^{s-1}(M)$ . Since  $S_\theta$  commutes with  $S$ ,  $T$  in the sense of distributions, it follows that the distributions  $Su - u_S$  and  $Tu - u_T \in H_q^{r-1}(M)$  are  $S_\theta$ -invariant. Let  $s(q) > 0$  be such that for all  $0 < s < s(q)$  and for almost all  $\theta \in S^1$  the space  $\mathcal{J}_{q\theta}^s(M)$  is 1-dimensional (see Theorem 4.27). If  $s(q) > 1 - r$ , since  $Su - u_S$  and  $Tu - u_T \in H_q^{r-1}(M)$  are  $S_\theta$ -invariant and orthogonal to constant functions, it follows that  $Su = u_S$  and  $Tu = u_T$ . The latter

identities imply that  $u \in H_q^r(M)$  and by (5.11) the required bound (5.1) is satisfied. The proof of the induction step is therefore completed.  $\square$

**5.2. Ergodic integrals.** Optimal results on the loss of regularity for almost all orientable quadratic differentials (and almost all directions) will be derived from bounds on ergodic integrals by the Gottschalk-Hedlund theorem. The required bounds will be proved along the lines of §9 in [For02] with the key improvement given by the ‘spectral gap’ Theorem 4.36.

The key idea of the argument given in [For02] consists in studying the dynamics of the distributional cocycle  $\{\Phi_t^1\}_{t \in \mathbb{R}}$  on the infinite dimensional (non-closed) sub-bundles  $\Gamma_\kappa^\pm \subset W_\kappa^{-1}(M)$  of 1-dimensional currents generated by segments of leaves of the horizontal and vertical foliations.

Let  $\mathcal{T} > 0$ . We will denote by  $\gamma_{\pm q}^\mathcal{T}$  a positively oriented segment of length  $\mathcal{T} > 0$  of a leaf of the measured foliation  $\mathcal{F}_{\pm q}$  respectively. By the trace theorems for standard Sobolev spaces and by the comparison Lemma 2.11, the vector spaces  $\Gamma_q^\pm$  generated by the sets  $\{\gamma_{\pm q}^\mathcal{T}(p) | (p, \mathcal{T}) \in M \times \mathbb{R}\}$  are subspaces of the weighted Sobolev space  $W_q^{-s}(M)$  of 1-dimensional currents for any  $s \geq 1$  and the corresponding sub-bundles  $\Gamma_\kappa^\pm$  are invariant under the action of the cocycle  $\{\Phi_t^s\}_{t \in \mathbb{R}}$ . Let  $F_q : M \times \mathbb{R} \rightarrow M$  [ $F_{-q} : M \times \mathbb{R} \rightarrow M$ ] be the (almost everywhere defined) flow of the horizontal [vertical] vector fields  $S_q$  [ $T_q$ ] on  $M$ . The horizontal and vertical foliations  $\mathcal{F}_{\pm q}$  coincide almost everywhere with the orbit foliations of the flows  $F_{\pm q}$ . Let  $\gamma_{\pm q}^\mathcal{T} \in \Gamma_q^\pm$  be a positively oriented segment with initial point  $p^\pm \in M$  and let  $\alpha := f^+ \eta_T + f^- \eta_S \in W_q^s(M)$ . The following identity holds:

$$(5.12) \quad \gamma_{\pm q}^\mathcal{T}(\alpha) = \int_0^\mathcal{T} f^\pm \circ F_{\pm q}(p^\pm, \tau) d\tau ,$$

The ergodic averages of the functions  $f^\pm \in H_q^s(M)$  (for  $s \geq 1$ ) can therefore be understood by studying the dynamics of the ‘renormalization’ cocycles  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  on the sub-bundle  $\Gamma_\kappa^\pm$ . In [For02] we have proved the following basic estimate on the weighted Sobolev norm of the currents  $\gamma_{\pm q}^\mathcal{T} \in W_q^{-1}(M)$ . Let  $R_q$  be the flat metric with conical singularities associated to the quadratic differential  $q \in \mathcal{M}_\kappa^{(1)}$  and let  $\|q\|$  denote the  $R_q$ -length of the shortest saddle connection of the flat surface  $(M, R_q)$ . (We recall that a *saddle connection* is a segment joining conical points).

**Lemma 5.2.** ([For02], Lemma 9.2) *There exists a constant  $K > 0$  such that, for all quadratic differentials  $q \in \mathcal{M}_\kappa^{(1)}$ ,*

$$(5.13) \quad |\gamma_{\pm q}^\mathcal{T}|_{W_q^{-1}(M)} \leq K(1 + \frac{\mathcal{T}}{\|q\|}) .$$



The above estimate is a ‘trivial’ bound, linear with respect to the length  $\mathcal{T} > 0$  of the orbit segment, and can be quite easily derived from the Sobolev trace theorem for rectangles in the euclidean plane  $\mathbb{R}^2$ . The number  $\|q\| > 0$ , which measures the pinching of the flat surface, gives the order of magnitude of the edges of the largest flat rectangle which can be embedded in the flat surface  $(M, R_q)$  around an arbitrary regular point.

We recall below the terminology and the notations, introduced in §9 of [For02], concerning return trajectories of translation flows.

**Definition 5.3.** A point  $p \in M$  is *regular* with respect to a measured foliation  $\mathcal{F}$  if it belongs to a regular (non-singular) leaf of  $\mathcal{F}$ . For any quadratic differential  $q \in \mathcal{Q}(M)$ . A point  $p \in M$  will be said to be *q-regular* if it is regular with respect to the horizontal and vertical foliations  $\mathcal{F}_{\pm q}$ .

The set of *q-regular* points is of full measure and it is equivariant under the actions of the group  $\text{Diff}^+(M)$  and of the Teichmüller flow on  $\mathcal{Q}(M)$ .

**Definition 5.4.** Let  $p \in M$  be a *q-regular* point and let  $I_{\pm q}(p)$  be the vertical [horizontal] segment of length  $\|q\|/2$  centered at  $p$ . A *forward horizontal [vertical] return time* of  $p \in M$  is a real number  $\mathcal{T} := \mathcal{T}_{\pm q}(p) > 0$  such that  $F_{\pm q}(p, \mathcal{T}) \in I_{\mp q}(p)$ . If  $\mathcal{T} > 0$  is a horizontal [vertical] return time for a *q-regular* point  $p \in M$ , the horizontal [vertical] forward orbit segment  $\gamma_{\pm q}^{\mathcal{T}}(p)$  will be called a *forward horizontal [vertical] return trajectory* at  $p$ .

There is a natural map from the set of horizontal [vertical] return trajectories into the set of homotopically non-trivial closed curves.

**Definition 5.5.** The *closing* of any horizontal [vertical] trajectory segment  $\gamma_{\pm q}^{\mathcal{T}}(p)$  is the piece-wise smooth homotopically non-trivial closed curve

$$(5.14) \quad \widehat{\gamma}_{\pm q}^{\mathcal{T}}(p) := \gamma_{\pm q}^{\mathcal{T}}(p) \cup \gamma(p, F_{\pm q}(p, \mathcal{T})) ,$$

obtained as the union of the trajectory segment  $\gamma_{\pm q}^{\mathcal{T}}(p)$  with the shortest geodesic segment  $\gamma(p, F_{\pm q}(p, \mathcal{T}))$  joining its endpoints  $p$  and  $F_{\pm q}(p, \mathcal{T})$ .

Let  $\mathcal{T}_{\pm q}^{(1)}(p) > 0$  be the *forward horizontal [vertical] first return time* of a *q-regular* point  $p \in M$ , defined to be the real number

$$(5.15) \quad \mathcal{T}_{\pm q}^{(1)}(p) := \min\{\mathcal{T} > 0 \mid F_{\pm q}(p, \mathcal{T}) \in I_{\mp q}(p)\} .$$

The corresponding forward horizontal [vertical] trajectory  $\gamma_{\pm q}^{(1)}(p)$  with initial point  $p$  will be called the *forward horizontal [vertical] first return trajectory* at  $p$ . The following bounds hold for first return times:

**Lemma 5.6.** ([For02], Lemma 9.2') *There exists a measurable function  $K_r : \mathcal{M}_\kappa \rightarrow \mathbb{R}^+$  such that, if  $\mathcal{T}_{\pm q}^{(1)}(p)$  is the forward horizontal [vertical]*

first return time of a  $q$ -regular point  $p \in M$ , then

$$(5.16) \quad \|q\|/2 \leq \mathcal{T}_{\pm q}^{(1)}(p) \leq K_r(q) .$$

The lower bound in (5.16) is an immediate consequence of the definition of  $\|q\|$  as the length of the shortest saddle connection, while the upper bound depends essentially on the fact that the first return map of a translation flow to a transverse interval is an interval exchange transformation, hence the return-time function is piece-wise constant (and bounded).

In §9.3 of [For02] we have constructed special sequences of horizontal [vertical] return times for almost all quadratic differentials generated. Such special return times are related to visiting times of the Teichmüller flow, which ‘renormalizes’ translation flows, to appropriate compact subsets of positive measure. Let  $q \in \mathcal{M}_\kappa^{(1)}$  be a Birkhoff generic point of the Teichmüller flow  $\{G_t\}_{t \in \mathbb{R}}$  and let  $\mathcal{S}_\kappa(q) \subset \mathcal{M}_\kappa^{(1)}$  be a smooth compact hypersurface of codimension 1, such that  $q \in \mathcal{S}_\kappa(q)$  and  $\mathcal{S}_\kappa(q)$  is transverse to the Teichmüller flow. Let  $(t_k)$  be the sequence of visiting times of the orbit  $\{G_t(q) \mid t \in \mathbb{R}\}$  to  $\mathcal{S}_\kappa(q)$ . Since, by definition, for any  $t \in \mathbb{R}$ ,

$$(5.17) \quad (\mathcal{F}_{G_t(q)}, \mathcal{F}_{-G_t(q)}) = (e^{-t}\mathcal{F}_q, e^t\mathcal{F}_{-q}) ,$$

if  $t := t_k < 0$  is a *backward* visiting time of the orbit  $\{G_t(q)\}_{t \in \mathbb{R}}$ , any forward first return trajectory of the horizontal foliation  $\mathcal{F}_{G_t(q)}$  is a forward return trajectory of the horizontal foliation  $\mathcal{F}_q$ , provided  $|t_k|$  is sufficiently large. In a similar way, if  $t := t_k > 0$  is a *forward* visiting time of  $\{G_t(q)\}_{t \in \mathbb{R}}$ , any forward first return trajectory of the vertical foliation  $\mathcal{F}_{-G_t(q)}$  is a forward return trajectory of the vertical foliation  $\mathcal{F}_{-q}$ .

By the *closing* of the return trajectories of the horizontal [vertical] foliation, as in (5.14), we obtain *closed* currents of Sobolev order 1. The evolution of such currents under the action of the Teichmüller flow is therefore described by the cocycles  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  on the bundle  $\mathcal{Z}_\kappa^s(M)$  for any  $s \geq 1$ . In the case  $s = 1$  the dynamics of the cocycle  $\{\Phi_t^1 | \mathcal{Z}_\kappa^1(M)\}$  is completely described by Theorem 4.30. In the following we will analyze Lyapunov exponents of closed currents given by the closing of horizontal [vertical] trajectories of translation flows under the cocycles  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  for  $s > 1$ . A complete description of the Lyapunov structure of the cocycles  $\{\Phi_t^s | \mathcal{Z}_\kappa^s(M)\}$  for  $s > 1$  is not relevant for our purposes and will not be attempted.

Let  $\mu$  be a  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic measure on a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials (in the sense of Definitions 4.13 and 4.4). We recall that according to [For02] all the canonical absolutely continuous,  $SL(2, \mathbb{R})$ -invariant measures on any connected component of any stratum of orientable quadratic differentials are KZ-hyperbolic. A different and stronger proof which establishes that all the afore-mentioned

measures are KZ-simple has been given in [AV05] (see Theorem 4.6). The latter theorem is not necessary for any of the results of this paper to hold.

By Theorem 4.30, for any  $s \geq 1$ , there exists a measurable splitting

$$(5.18) \quad \mathcal{Z}_\kappa^s(M) = \mathcal{B}_{\kappa,+}^1(M) \oplus \mathcal{B}_{\kappa,-}^1(M) \oplus \mathcal{E}_\kappa^s(M).$$

The Lyapunov exponents of the restriction of the distributional cocycle  $\{\Phi_t^s\}_{t \in \mathbb{R}}$  to the finite dimensional sub-bundles  $\mathcal{B}_{\kappa,+}^1(M)$  [ $\mathcal{B}_{\kappa,-}^1(M)$ ] are equal to the non-negative [non-positive] exponents of the Kontsevich-Zorich cocycle. It follows from the non-uniform hyperbolicity hypothesis for the latter that all Lyapunov exponents of the cocycle  $\{\Phi_t^s|_{\mathcal{B}_{\kappa,+}^1(M)}\}$  are strictly positive while all Lyapunov exponents of the cocycle  $\{\Phi_t^s|_{\mathcal{B}_{\kappa,-}^1(M)}\}$  are strictly negative. In particular, by Oseledec's theorem there exists a measurable function  $\Lambda_\kappa^s : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  and a real number  $\alpha_\mu > 0$  such that, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , for all  $C \in \mathcal{B}_{-q}^1(M)$  and all  $t \geq 0$ ,

$$(5.19) \quad |\Phi_t^s(C)|_{-s} \leq \Lambda_\kappa^s(q) |C|_{-s} e^{-\alpha_\mu t}.$$

In [For02], we have proved that for  $s = 1$  the restriction  $\{\Phi_t^s|_{\mathcal{E}_\kappa^s(M)}\}$  has 0 as the unique Lyapunov exponent. In fact, this cocycle is isometric with respect to a continuous Lyapunov norm (see Theorem 4.30). For  $s > 1$ , estimates on Lyapunov exponents and Lyapunov norms for the restriction  $\Phi_t^s|_{\mathcal{E}_\kappa^s(M)}$  will be derived from Theorem 4.36. Let  $\sigma := s - 1 > 0$ . For any  $q \in \mathcal{M}_\kappa^{(1)}$ , any  $l < 1$  and any  $\epsilon > 0$ , let

$$(5.20) \quad \mathcal{N}_\kappa^{\sigma,\epsilon,l}(q) := \sup_{U \in \mathcal{H}_q^0(M)} \frac{\mathcal{N}_q^{\sigma,-\epsilon}(U)}{|U|_0 + \mathcal{N}_q^{1,l}(S_q U)}.$$

By Theorem 4.36 for any  $l < 1$  there exists  $\epsilon := \epsilon(\sigma, l) > 0$  such that formula (5.20) defines a (measurable) function  $\mathcal{N}_\kappa^{\sigma,\epsilon,l} : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$ .

Let  $\Pi_{\pm q} : \mathcal{Z}_q^1(M) \rightarrow \mathcal{B}_{\pm q}^1(M)$  and  $\mathcal{E}_q : \mathcal{Z}_q^1(M) \rightarrow \mathcal{E}_q^1(M)$  be the projections determined by the splitting (5.18). For any  $s \geq 1$ , the restrictions of the projections  $\Pi_{\pm q}$  and  $\mathcal{E}_q$  to the subspace  $\mathcal{Z}_q^s(M) \subset \mathcal{Z}_q^1(M)$  can be extended to projections  $\Pi_{\pm q}^s : W_q^{-s}(M) \rightarrow \mathcal{B}_{\pm q}^1(M)$  and  $\mathcal{E}_q^s : W_q^{-s}(M) \rightarrow \mathcal{E}_q^s(M)$ , defined on the Sobolev space  $W_q^{-s}(M)$  of 1-dimensional currents, by composition with the orthogonal projection  $W_q^{-s}(M) \rightarrow \mathcal{Z}_q^s(M)$  onto the closed subspace of closed currents. Let  $d_\kappa^s : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  be the *distorsion* of the splitting (5.18), that is, the function defined for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  as

$$(5.21) \quad d_\kappa^s(q) := \sup_{C \in \mathcal{Z}_q^s(M)} \frac{|\Pi_q(C)|_{-s} + |\Pi_{-q}(C)|_{-s} + |\mathcal{E}_q(C)|_{-s}}{|C|_{-s}}.$$

Let  $s > 1$  and  $l < 1$ . Let  $\mathcal{P}_\kappa^{(1)} \subset \mathcal{M}_\kappa^{(1)}$  be a compact set satisfying the

following conditions:

- (1) All  $q \in \mathcal{P}_\kappa^{(1)}$  are Birkhoff generic points for the Teichmüller flow  $\{G_t\}$  and Oseledec regular points for the cocycle  $\{\Phi_t^s | \mathcal{B}_\kappa^1(M)\}$ ;
- (2) the set  $\mathcal{P}_\kappa^{(1)}$  is transverse to the Teichmüller flow and has positive transverse measure;
- (3) there exists a constant  $K_r^{(1)} > 0$  such that, for all  $q \in \mathcal{P}_\kappa^{(1)}$  and all  $q$ -regular  $p \in M$ , the first return times  $\mathcal{T}_{\pm q}^{(1)}(p) \leq K_r^{(1)}$ ;
- (4) the real-valued functions  $\Lambda_\kappa^s$ ,  $\mathcal{N}_\kappa^{\sigma, \epsilon, l}$  and  $\delta_\kappa^1$ , introduced respectively in formulas (5.19), (5.20) and (5.21), are bounded on  $\mathcal{P}_\kappa^{(1)}$ .

By the ergodicity of the system  $(\{G_t\}, \mu)$ , by the Oseledec's theorem for the cocycle  $\Phi_t^s | \mathcal{B}_\kappa^1(M)$  and by Lemma 4.6, it follows that the union of all sets  $\mathcal{P}_\kappa^{(1)}$  with the properties (1) – (4) is a full measure subset of  $\mathcal{M}_\kappa^{(1)}$ .

**Definition 5.7.** Let  $q \in \mathcal{M}_\kappa^{(1)}$  and  $(t_k)_{k \in \mathbb{N}}$  be the sequence of visiting times of the *backward* orbit  $\{G_t(q) | t \leq t_1 = 0\}$  to a compact positive measure set  $\mathcal{P}_\kappa^{(1)} \subset \mathcal{M}_\kappa^{(1)}$ . A *principal sequence*  $(\mathcal{T}_q^{(k)}(p))_{k \in \mathbb{N}}$  of forward return times for the horizontal foliation  $\mathcal{F}_q$  at a  $q$ -regular point  $p \in M$  is the sequence

$$(5.22) \quad \mathcal{T}_q^{(k)}(p) := \mathcal{T}_{G_t(q)}^{(1)}(p) \exp |t|, \quad t = t_k.$$

For each  $k \in \mathbb{N}$ , a *horizontal principal (forward) return trajectory*  $\gamma_q^{(k)}(p)$  at a  $q$ -regular point  $p \in M$  is the horizontal forward return trajectory at the point  $p$  corresponding to a principal return time  $\mathcal{T}_q^{(k)}(p) > 0$ .

We remark that a horizontal principal return trajectory  $\gamma_q^{(k)}(p)$  coincides with the horizontal *first* return trajectory at  $p$  of the quadratic differential  $G_t(q)$ ,  $t = t_k < 0$ . A similar definition can be given for the vertical foliation  $\mathcal{F}_{-q}$  by considering forward visiting times of the Teichmüller flow.

The following standard splitting lemma allows to reduce the analysis of arbitrary regular trajectories to that of principal return trajectories.

**Lemma 5.8.** ([For02], Lemma 9.4) *Under conditions (1) – (3) there exists a constant  $K_{\mathcal{P}} > 0$  such that the following holds. Let  $q \in \mathcal{P}_\kappa^{(1)}$  and, for any  $\mathcal{T} > 0$ , let  $\gamma_q^{\mathcal{T}}(p)$  be a forward trajectory at a  $q$ -regular point  $p \in M$ . There exists a finite set of points  $\{p_j^{(k)} | 1 \leq k \leq n, 1 \leq j \leq m_k\} \subset \gamma_q^{\mathcal{T}}(p)$ , such that the principal return trajectories  $\gamma_q^{(k)}(p_j^{(k)}) \subset \gamma_q^{\mathcal{T}}(p)$  do not overlap and*

$$(5.23) \quad \gamma_q^{\mathcal{T}}(p) = \sum_{k=1}^n \sum_{j=1}^{m_k} \gamma_q^{(k)}(p_j^{(k)}) + b_q^{\mathcal{T}}(p),$$

with  $m_k \leq K_{\mathcal{P}} e^{|t_{k+1}| - |t_k|}$ , for  $1 \leq k \leq n$ ,

and the  $R_q$ -length  $L_q(b_q^{\mathcal{T}}(p)) \leq K_{\mathcal{P}}$ .

*Proof.* We recall the proof given in [For02], Lemma 9.4. The argument is based on the following estimate on principal return times. By (5.22), Lemma 5.6 and condition (3), there exists a constant  $K_{pr} > 0$  such that, for all  $q \in \mathcal{P}_\kappa^{(1)}$ , all  $q$ -regular points  $p \in M$  and all  $k \in \mathbb{N}$ ,

$$(5.24) \quad K_{pr}^{-1} \exp |t_k| \leq \mathcal{T}_q^{(k)}(p) \leq K_{pr} \exp |t_k| .$$

Let  $n = \max\{k \in \mathbb{N} \mid \mathcal{T}_q^{(k)}(p) \leq \mathcal{T}\}$ . The maximum exists (finite) by (5.24). Let  $p_1^{(n)} := p$ . The sequence  $(p_j^{(k)})$  with the properties stated in (5.23) can be constructed by a finite iteration of the following procedure. Let  $p_j^{(k)}$  be the last point already determined in the sequence and let

$$(5.25) \quad p_{j+}^{(k)} := F_q(p_j^{(k)}, \mathcal{T}_q^{(k)}(p_j^{(k)})) \in \gamma_q^{\mathcal{T}}(p) .$$

Let then  $k' \in \{1, \dots, k\}$  be the largest integer such that

$$(5.26) \quad F_q(p_{j+}^{(k)}, \mathcal{T}_q^{(k')}(p_{j+}^{(k)})) \in \gamma_q^{\mathcal{T}}(p) .$$

If  $k' = k$ , let  $p_{j+1}^{(k)} := p_{j+}^{(k)}$ . If  $k' < k$ , let  $m_k := j$ ,  $m_h = 0$  (no points) for all  $k' < h < k$  and  $p_1^{(k')} := p_{j+}^{(k)}$ . The iteration step is concluded. By (5.24) it follows that

$$(5.27) \quad K_{pr}^{-1} e^{|t_k|} m_k \leq \mathcal{T}_q^{(k+1)}(p_1^{(k)}) \leq K_{pr} e^{|t_{k+1}|} .$$

The length of the remainder  $b_q^{\mathcal{T}}(p)$  has to be less than any upper bound for the length of first return times. Hence, by (5.27), Lemma 5.6 and condition (3), the estimates in (5.23) are proved and the argument is concluded.  $\square$

The result below gives a fundamental uniform estimate on ergodic integrals.

**Theorem 5.9.** *Let  $\mu$  be an  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic measure on a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials. For any  $s > 1$  there exists a measurable function  $\Gamma_\kappa^s : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that, for  $\mu$ -almost  $q \in \mathcal{M}_\kappa^{(1)}$ , for all  $q$ -regular  $p \in M$  and for all  $\mathcal{T} > 0$ ,*

$$(5.28) \quad |\Pi_{-q}^s(\gamma_q^{\mathcal{T}}(p))|_{-s} + |\mathcal{E}_q^s(\gamma_q^{\mathcal{T}}(p))|_{-s} \leq \Gamma_\kappa^s(q) .$$

*Proof.* Let  $\mathcal{P} := \mathcal{P}_\kappa^{(1)}$  be a compact set satisfying conditions (1) – (4) listed above. For  $\mu$ -almost all  $q \in \mathcal{P}$ , there exists a sequence  $(t_k)_{k \in \mathbb{N}}$  of backward return times of the Teichmüller orbit  $G_t(q)$  to the set  $\mathcal{P}$ . By Lemma 5.8 the uniform estimate (5.28) can be reduced (exponential) estimates for principal return trajectories  $\gamma_q^{(k)}(p)$ . We claim that for every  $s > 1$  there exist constants  $K_{\mathcal{P}}^s > 0$  and  $\alpha_s > 0$  such that for  $\mu$ -almost all  $q \in \mathcal{P}$ , for all  $q$ -regular  $p \in M$  and all  $k \in \mathbb{N}$ ,

$$(5.29) \quad |\Pi_{-q}^s(\gamma_q^{(k)}(p))|_{-s} + |\mathcal{E}_{-q}^s(\gamma_q^{(k)}(p))|_{-s} \leq K_{\mathcal{P}}^s e^{-\alpha_s |t_k|} .$$

By Lemma 5.2, by conditions (3) and (4) on the set  $\mathcal{P}$ , in particular the bound on the distortion, there exists a constant  $K_1^s(\mathcal{P}) > 0$  such that the following holds. For  $\mu$ -almost all  $q \in \mathcal{P}$  and all  $q$ -regular  $p \in M$ , the closing  $\widehat{\gamma}_q^{(1)}(p)$  of the first return horizontal trajectory  $\gamma_q^{(1)}(p)$  (see Definition 5.5) satisfies the following bound:

$$(5.30) \quad |\Pi_{-q}(\widehat{\gamma}_q^{(1)}(p))|_{-1} + |\mathcal{E}_q(\widehat{\gamma}_q^{(1)}(p))|_{-1} \leq K_1(\mathcal{P})$$

Since by definition of the principal return trajectories

$$(5.31) \quad \widehat{\gamma}_q^{(k)}(p) = \Phi_{|t|}^s(\widehat{\gamma}_{G_t(q)}^{(1)}(p)) \ , \quad \text{for any } t = t_k \leq 0 \ ,$$

it follows from the initial bound (5.30), from the invariance of the sub-bundle  $\mathcal{B}_{\kappa,-}^1(M)$  under the cocycles  $\{\Phi_t^s\}$ , from the Lyapunov bound (5.19) on the cocycle  $\{\Phi_t^s|\mathcal{B}_{\kappa,-}^1(M)\}$  and from condition (4) on the set  $\mathcal{P} \subset \mathcal{M}_\kappa^{(1)}$  that there exists a constant  $K_2^s(\mathcal{P}) > 0$  such that the following estimate holds. For  $\mu$ -almost all  $q \in \mathcal{P}$  and for all  $q$ -regular  $p \in M$ ,

$$(5.32) \quad |\Pi_{-q}(\widehat{\gamma}_q^{(k)}(p))|_{-s} \leq K_2^s(\mathcal{P}) e^{-\alpha_\mu |t_k|} \ , \quad \text{for all } k \in \mathbb{N} \ .$$

A similar estimate holds for the projections of principal return trajectories on the sub-bundle  $\mathcal{E}_\kappa^1(M)$  of exact currents. In fact, there exist  $K_3^s(\mathcal{P}) > 0$  and  $\epsilon_s > 0$  such that, for  $\mu$ -almost all  $q \in \mathcal{P}$  and for all  $q$ -regular  $p \in M$ ,

$$(5.33) \quad |\mathcal{E}_q(\widehat{\gamma}_q^{(k)}(p))|_{-s} \leq K_2^s(\mathcal{P}) e^{-\epsilon_s |t_k|} \ , \quad \text{for all } k \in \mathbb{N} \ .$$

The estimate (5.33) is proved as follows. By definition of the bundle  $\mathcal{E}_\kappa^1(M)$  of exact currents, for all  $q \in \mathcal{M}_\kappa^{(1)}$  and all  $q$ -regular  $p \in M$ , there exists a unique function  $U_q^{(1)}(p) \in \mathcal{H}_q^0(M) \subset L_q^2(M)$  such that

$$(5.34) \quad \mathcal{E}_q(\widehat{\gamma}_q^{(1)}(p)) = dU_q^{(1)}(p) \ .$$

By the definitions of the distributional cocycles  $\{\Phi_t^s\}$  and  $\{G_t^s\}$  the following identity of cocycles holds for any  $s \geq 1$ :

$$(5.35) \quad \{\Phi_t^s \circ d\} = \{d \circ G_t^{s-1}\} \quad \text{on } H_q^{-s+1}(M) \ .$$

The exponential estimate (5.33) will therefore follow from Theorem 4.36 if we can prove that there exist a positive number  $l < 1$  and a constant  $K_3(\mathcal{P}) > 0$  such that, for  $\mu$ -almost all  $q \in \mathcal{P}$  and all  $q$ -regular  $p \in M$ , the following holds:

$$(5.36) \quad |U_q^{(1)}(p)|_0 + \mathcal{N}_q^{1,l}[S_q U_q^{(1)}(p)] \leq K_3^s(\mathcal{P}) \ .$$

Let  $|\cdot|_\kappa$  be the norm on the bundle  $\mathcal{E}_\kappa^1(M)$  of exact currents defined as follows: for any  $q \in \mathcal{M}_\kappa^{(1)}$  and  $C \in \mathcal{E}_q^1(M)$ ,

$$(5.37) \quad |C|_\mathcal{E} := |U|_{L_q^2(M)} \ , \quad \text{if } C = dU \text{ with } U \in \mathcal{H}_q^0(M) \ .$$

By the definitions of weighted Sobolev norms, it is immediate to prove that the map  $d : \mathcal{H}_q^0(M) \rightarrow \mathcal{E}_q^1(M)$  is a continuous bijective map of Hilbert spaces. Hence, by definition (5.37) and by the open mapping theorem, for each  $q \in \mathcal{M}_\kappa^{(1)}$  there exists  $K(q) > 0$  such that

$$(5.38) \quad K(q) |\cdot|_\varepsilon \leq |\cdot|_{-1} \leq |\cdot|_\varepsilon \quad \text{on } \mathcal{E}_q^1(M).$$

Since the sub-bundle  $\mathcal{H}_\kappa^0(M) \subset H_\kappa^0(M)$  and the norms  $|\cdot|_\varepsilon, |\cdot|_{-1}$  on the bundle  $\mathcal{E}_\kappa^1(M)$  are all continuous, the function  $K : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  can be chosen continuous (see Lemma 9.3 in [For02] for a detailed argument). Hence by the estimates (5.30) and by identity (5.34), there exists a constant  $C_1(\mathcal{P}) > 0$  such that, for  $\mu$ -almost all  $q \in \mathcal{P}$  and all  $q$ -regular  $p \in M$ ,

$$(5.39) \quad |U_q^{(1)}(p)|_{L_q^2(M)} \leq K(q)^{-1} |\mathcal{E}_q(\widehat{\gamma}_q^{(1)}(p))|_{-1} \leq C_1(\mathcal{P}).$$

The proof of the estimate (5.36) is completed as follows. Since  $\gamma_q^{(1)}(p)$  is a horizontal trajectory, by the splitting (5.18) and by the identity (5.34), the following formula holds:

$$(5.40) \quad S_q U_q^{(1)}(p) = \iota_{S_q} [\widehat{\gamma}_q^{(1)}(p) - \gamma_q^{(1)}(p)] - \iota_{S_q} \Pi_{-q} [\widehat{\gamma}_q^{(1)}(p)].$$

Since the distribution  $\iota_{S_q} [\widehat{\gamma}_q^{(1)}(p) - \gamma_q^{(1)}(p)]$  is given by integration along a vertical arc of unit length, by the Sobolev embedding Lemma 5.2 and by the logarithmic law of geodesics for the Teichmüller geodesic flow on moduli spaces (see [Mas93], Prop. 1.2, or §4.1, formula (4.49)) it follows that for any  $l > 0$  there exists a constant  $C_l > 0$  such that, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and all  $q$ -regular  $p \in M$ , the Lyapunov norm

$$(5.41) \quad \mathcal{N}_q^{1,l} (\iota_{S_q} [\widehat{\gamma}_q^{(1)}(p) - \gamma_q^{(1)}(p)]) \leq C_l.$$

By the cocycle isomorphism (4.78), by the Lyapunov estimate (5.19) and by condition (4) on the set  $\mathcal{P}$ , since the distribution  $\iota_{S_q} \Pi_{-q} [\widehat{\gamma}_q^{(1)}(p)] \in \mathcal{J}_{-q}^1(M)$ , there exists a constant  $C_2(\mathcal{P}) > 0$  such that, for any  $1 > l \geq 1 - \alpha_\mu$ , for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and all  $q$ -regular  $p \in M$ , the Lyapunov norm

$$(5.42) \quad \mathcal{N}_q^{1,l} (\iota_{S_q} \Pi_{-q} [\widehat{\gamma}_q^{(1)}(p)]) \leq C_2(\mathcal{P}).$$

The bound (5.36) follows immediately from the bound (5.39), from the identity (5.40) and from the bounds (5.41) and (5.42).

As hinted above, the estimate (5.36) implies the required exponential estimate (5.33) on the projections of principal return trajectories on the space of exact currents. In fact, by Theorem 4.36 and by condition (4) on the set  $\mathcal{P}$ , for any  $1 > l \geq 1 - \alpha_\mu$  and any  $\sigma > 0$ , there exist  $K_4^\sigma(\mathcal{P}) > 0$  and  $\epsilon_s := \epsilon(l, \sigma) > 0$  such that, for  $\mu$ -almost all  $q \in \mathcal{P}$  and all  $q$ -regular  $p \in M$ ,

$$(5.43) \quad \mathcal{N}_q^{\sigma, \epsilon} (U_q^{(1)}(p)) \leq K_4^s(\mathcal{P}).$$

Let us introduce the following notation: for any  $k \in \mathbb{N}$ , let

$$(5.44) \quad q_k := G_{t_k}(q) \in \mathcal{P} \quad \text{and} \quad U_q^{(k)}(p) := G_{|t_k|}^0(U_{q_k}^{(1)}(p)) \in L_q^2(M).$$

Since the splittings (5.18) are  $\{\Phi_t^s\}$ -invariant (in particular for  $s = 1$ ), by the definition of principal return trajectories (5.31), by the identity (5.34) and the cocycle identity (5.35), the following holds:

$$(5.45) \quad \begin{aligned} \mathcal{E}_q(\hat{\gamma}_q^{(k)}(p)) &= \mathcal{E}_q \circ \Phi_{|t_k|}^1(\hat{\gamma}_{q_k}^{(1)}(p)) = \\ &= \Phi_{|t_k|}^1(dU_{q_k}^{(1)}(p)) = dU_q^{(k)}(p) \end{aligned}$$

Let  $s > 1$  and  $\sigma := s - 1$ . It follows from (5.43) (by the definition (4.107) of Lyapunov norms) that, for  $\mu$ -almost all  $q \in \mathcal{P}$  and all  $q$ -regular  $p \in M$ ,

$$(5.46) \quad |\mathcal{E}_q(\hat{\gamma}_q^{(k)}(p))|_{-s} \leq |G_{|t_k|}^0(U_{q_k}^{(1)}(p))|_{-\sigma} \leq K_4^s(\mathcal{P}) e^{-\epsilon_s |t_k|}.$$

The crucial exponential estimate (5.29) on projections of principal return trajectories, claimed above, can now be derived from estimates (5.32) and (5.33) together with the remark that, for any  $l > 0$ , there exists a constant  $C_l > 0$  such that, for  $\mu$ -almost all  $q \in \mathcal{P}$  and all  $q$ -regular  $p \in M$ ,

$$(5.47) \quad |\gamma_q^{(k)}(p) - \hat{\gamma}_q^{(k)}(p)|_{-1} \leq C_l e^{-l|t_k|}.$$

By the definition of principal return trajectories and closing, the latter estimate follows immediately from (5.41), since  $\gamma_q^{(k)}(p) - \hat{\gamma}_q^{(k)}(p)$  are currents of integration along a vertical  $q$ -regular arc.

The required estimate (5.28) finally follows from estimate (5.29) by the trajectory splitting Lemma 5.8. In fact, for any  $s > 1$  there exist positive real numbers  $\omega_s^{(1)} < \omega_s^{(2)}$  and a measurable map  $k_s : \mathcal{P} \rightarrow \mathbb{N}$  such that  $\alpha_s \omega_s^{(1)} > \omega_s^{(2)} - \omega_s^{(1)}$  and, for  $\mu$ -almost all  $q \in \mathcal{P}$ ,

$$(5.48) \quad \omega_s^{(1)} k \leq |t_k(q)| \leq \omega_s^{(2)} k, \quad \text{for all } k \geq k_s(q).$$

Let  $\{p_j^{(k)} | 1 \leq k \leq n, 1 \leq j \leq m_k\} \subset \gamma^{\mathcal{T}}(p)$  be the sequence of  $q$ -regular points constructed in Lemma 5.8. Since  $\alpha_s \omega_s^{(1)} - (\omega_s^{(2)} - \omega_s^{(1)}) > 0$ , by estimates (5.29) and (5.48), there exists a constant  $K_5^s(\mathcal{P}) > 0$  such that, for  $\mu$ -almost all  $q \in \mathcal{P}$ , all  $q$ -regular  $p \in M$  and all  $\mathcal{T} > 0$ ,

$$(5.49) \quad \begin{aligned} \sum_{k=k_s(q)}^n e^{|t_{k+1}| - |t_k|} |\Pi_{-q}^s(\gamma_q^{(k)}(p_j^{(k)}))|_{-s} &\leq K_5^s(\mathcal{P}); \\ \sum_{k=k_s(q)}^n e^{|t_{k+1}| - |t_k|} |\mathcal{E}_q^s(\gamma_q^{(k)}(p_j^{(k)}))|_{-s} &\leq K_5^s(\mathcal{P}). \end{aligned}$$



In addition, by Lemma 5.8 there exists a constant  $K_6(\mathcal{P}) > 0$  such that

$$(5.50) \quad L_q \left[ \sum_{k=1}^{k_s(q)} \sum_{j=1}^{m_k} \gamma_q^{(k)}(p_k^{(j)}) \right] \leq K_6(\mathcal{P}) \sum_{k=1}^{k_s(q)} e^{|t_{k+1}(q)|},$$

hence the estimate (5.28) follow from estimates (5.49), from the trajectory splitting Lemma 5.8, from estimate (5.50) and finally from Lemma 5.2, which yields a bound on weighted Sobolev norms of (horizontal and vertical) trajectories in terms of their  $R_q$ -length.  $\square$

**5.3. The generic case.** The above Theorem 5.9 implies, by a standard Gottschalk-Hedlund argument, the following sharp result on the existence of a *Green operator* for the (horizontal) cohomological equation in the case of generic orientable quadratic differentials.

**Theorem 5.10.** *For any  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic measure  $\mu$  on a stratum  $\mathcal{M}_\kappa^{(1)}$  of orientable quadratic differentials and for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , there exists a densely defined, anti-symmetric Green operator  $\mathcal{U}_q : \mathcal{H}_q^0(M) \rightarrow \mathcal{H}_q^0(M)$  for the horizontal cohomological equation  $S_q u = f$ . For any  $s > 1$ , the maximal domain of the Green operator  $\mathcal{U}_q$  on the Hilbert space  $\mathcal{H}_q^0(M) \subset L_q^2(M)$  contains the dense subspace*

$$(5.51) \quad [\mathcal{I}_q^s(M)]^\perp := \{f \in H_q^s(M) \mid \mathcal{D}(f) = 0 \text{ for all } \mathcal{D} \in \mathcal{I}_q^s(M)\},$$

and there exists a measurable function  $C_\kappa^s : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that the following holds. For any  $f \in [\mathcal{I}_q^s(M)]^\perp$ , the Green solution  $\mathcal{U}_q(f) \in L^\infty(M)$  has zero-average and satisfies the estimate:

$$(5.52) \quad |\mathcal{U}_q(f)|_{L^\infty(M)} \leq C_\kappa^s(q) |f|_s.$$

*Proof.* Let  $\{u_\mathcal{T}\}_{\mathcal{T} \in \mathbb{R}}$  be the 1-parameter family of measurable functions defined almost everywhere on  $M$  as follows: for all  $q$ -regular  $p \in M$ ,

$$(5.53) \quad u_\mathcal{T}(p) := \frac{1}{\mathcal{T}} \int_0^\mathcal{T} \int_0^\tau f \circ F_q(p, s) ds d\tau.$$

By Theorem 5.9 (and Lemma 5.2) for any  $s > 1$  there exists a measurable function  $C_\kappa^s : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , for all  $q$ -regular  $p \in M$  and all  $\mathcal{T} > 0$ ,

$$(5.54) \quad |\gamma_q^\mathcal{T}(p) - \Pi_q^s(\gamma_q^\mathcal{T}(p))|_{-s} \leq C_\kappa^s(q).$$

We recall that the projection  $\Pi_q^s : W_q^{-s}(M) \rightarrow \mathcal{B}_q^1(M)$  is defined as the composition of the orthogonal projection  $W_q^{-s}(M) \rightarrow \mathcal{Z}_q^s(M)$  with the projection  $\mathcal{Z}_q^s(M) \rightarrow \mathcal{B}_q^1(M)$  determined by the splitting (5.18).

If  $f \in [\mathcal{J}^s(M)]^\perp$ , the 1-form  $\alpha_f := f\eta_T \in W_q^s(M)$  is such that  $C(\alpha) = 0$  for any horizontally basic current  $C \in \mathcal{B}_q^1(M)$ . It follows from the estimate (5.54) that, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , for all  $q$ -regular  $p \in M$  and all  $\mathcal{T} > 0$ ,

$$(5.55) \quad |u_{\mathcal{T}}(p)| \leq |\gamma_q^{\mathcal{T}}(\alpha)| = |\{\gamma_q^{\mathcal{T}}(p) - \Pi_q^s(\gamma_q^{\mathcal{T}}(p))\}(\alpha)| \leq C_\kappa^s(q) |f|_s,$$

hence the family  $\{u_{\mathcal{T}}\}_{\mathcal{T} \in \mathbb{R}}$  is uniformly bounded in the Hilbert space  $L_q^2(M)$  for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ . In addition, a computation shows that if the horizontal flow  $F_q$  is ergodic, as  $\mathcal{T} \rightarrow +\infty$ ,

$$(5.56) \quad S_q u_{\mathcal{T}} = f - \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} f \circ F_q(\cdot, \tau) d\tau \rightarrow f \quad \text{in } L_q^2(M),$$

hence any weak limit  $u \in L_q^2(M)$  of the family  $\{u_{\mathcal{T}}\}_{\mathcal{T} \in \mathbb{R}}$  is a solution of the cohomological equation  $S_q u = f$ . Since any function  $f \in [\mathcal{J}^s(M)]^\perp$  has zero average, it follows by the definition (5.53) that  $u_{\mathcal{T}}$  has zero average for all  $\mathcal{T} > 0$ , hence any weak limit of the family  $\{u_{\mathcal{T}}\}_{\mathcal{T} \in \mathbb{R}}$  has zero average. By the ergodicity of the horizontal flow  $F_q$ , the cohomological equation  $S_q u = f$  has a unique zero average solution in  $\mathcal{U}_q(f) \in L_q^2(M)$ . It follows that the operator  $\mathcal{U}_q : [\mathcal{J}_q^s(M)]^\perp \rightarrow \mathcal{H}_q^0(M) \subset L_q^2(M)$  is well-defined and linear, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ . In addition, by the uniform bound (5.55) the function  $\mathcal{U}_q(f) \in L^\infty(M)$  and satisfies the required bound (5.52).

It remains to be proven that the linear operator  $\mathcal{U}_q : \mathcal{H}_q^0(M) \rightarrow \mathcal{H}_q^0(M)$  is anti-symmetric. For any  $f \in [\mathcal{J}_q^s(M)]^\perp$  and  $v \in \mathcal{H}_q^{s+1}(M) \subset H_q^{s+1}(M)$ , since  $\mathcal{U}_q(f) \in L_q^2(M)$  is a weak-solution of the equation  $S_q u = f$ ,

$$(5.57) \quad \langle \mathcal{U}_q(f), S_q v \rangle_q = -\langle f, v \rangle_q = -\langle f, \mathcal{U}_q(S_q v) \rangle_q.$$

Since the subspace  $\{S_q v \in H_q^s(M) | v \in H_q^{s+1}(M)\}$  is dense in the subspace  $[\mathcal{J}_q^s(M)]^\perp \subset H_q^s(M)$  and  $\mathcal{U}_q : [\mathcal{J}_q^s(M)]^\perp \rightarrow L_q^2(M)$  is continuous, it follows from identity (5.57) by a density argument that

$$(5.58) \quad \langle \mathcal{U}_q(f), g \rangle_q = -\langle f, \mathcal{U}_q(g) \rangle_q, \quad \text{for all } f, g \in [\mathcal{J}_q^s(M)]^\perp.$$

□

Optimal results on the regularity of solutions of the cohomological equation for intermediate ‘fractional’ regularity require smoothing and interpolation techniques in the presence of distributional obstructions. The following definition appears to capture the relevant condition on the obstructions which allow for effective smoothing and interpolation results.

**Definition 5.11.** Let  $\{H^s | s \geq 0\}$  be a 1-parameter family of normed spaces such that the following embeddings hold:

$$(5.59) \quad H^\infty := \bigcap_{s>0} H^s \subset H^s \subset H^r \subset H^0, \quad \text{for all } s \geq r.$$

The *order*  $\mathcal{O}^H(\mathcal{D})$  (with respect to  $\{H^s | s \geq 0\}$ ) of any linear functional  $\mathcal{D} \in (H^\infty)^*$  is the non-negative real number

$$(5.60) \quad \mathcal{O}^H(\mathcal{D}) := \inf\{s \geq 0 \mid \mathcal{D} \in (H^s)^*\}.$$

A finite system of linear functionals  $\{\mathcal{D}_1, \dots, \mathcal{D}_J\} \subset (H^\sigma)^*$  will be called  *$\sigma$ -regular* (with respect to the family  $\{H^s | s \geq 0\}$ ) if, for any  $\tau \in (0, 1]$  there exists a *dual* system  $\{u_1(\tau), \dots, u_J(\tau)\} \subset H^\sigma$  such that the following estimates hold. For all  $0 \leq r \leq \sigma$  and all  $\epsilon > 0$ , there exists a constant  $C_r^\sigma(\epsilon) > 0$  such that, for all  $i, j \in \{1, \dots, J\}$ ,

$$(5.61) \quad |u_j(\tau)|_r \leq C_r^\sigma(\epsilon) \tau^{\mathcal{O}^H(\mathcal{D}_j) - r - \epsilon}.$$

A finite system  $\{\mathcal{D}_1, \dots, \mathcal{D}_J\} \subset (H^s)^*$  will be called *regular* if it is  $\sigma$ -regular for any  $\sigma \geq s$ . A finite dimensional subspace  $\mathcal{J} \subset (H^\infty)^*$  will be called  *$\sigma$ -regular [regular]* if it admits a  $\sigma$ -regular [regular] basis.

It was proved in §1.4, in particular in Corollary 2.19, that for any  $s \geq 0$  the closure  $\overline{H_q^s(M)} \subset \bar{H}_q^s(M)$  of the weighted Sobolev space as a subspace of the *Friedrichs* weighted Sobolev space, is equal to the perpendicular of a subspace  $\mathcal{D}_q^s \subset \bar{H}_q^{-s}(M)$ , introduced in (2.94), of distributions supported on the singular set  $\Sigma_q \subset M$ . It follows by Theorem 2.17 and by Corollary 2.19 that all the spaces  $\mathcal{D}_q^s$  are regular with respect to the family  $\{\bar{H}_q^s(M) | s \geq 0\}$  of Friedrichs weighted Sobolev spaces. This regularity result is the key to the existence of smoothing operators for the family  $\{H_q^s(M) | s \geq 0\}$  of weighted Sobolev spaces. In fact, their construction in Theorem 2.24 is based on the regularity of the subspaces  $\mathcal{D}_q^s \subset \bar{H}_q^{-s}(M)$  and on the existence of smoothings for the family of Friedrichs weighted Sobolev spaces, which can be defined by appropriate truncations of the Fourier expansion.

Our goal in this section is to implement a similar strategy in order to construct smoothing families with values in the perpendicular of the subspaces of invariant distributions. We prove below, under the hypothesis that the Kontsevich-Zorich cocycle is non-uniformly hyperbolic, a preliminary result on the regularity of the spaces of horizontal [vertical] invariant distributions with respect to the family of weighted Sobolev spaces. The basic criterion for the regularity of spaces in invariant distributions is based on Theorem 4.35 and the following notion:

**Definition 5.12.** Let  $\mu$  be any  $G_t$ -invariant ergodic probability measure on any stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable holomorphic quadratic differentials. For any  $q \in \mathcal{R}_\mu$ , a simple invariant distribution  $\mathcal{D}^\pm \in \mathcal{I}_{\pm q}(M \setminus \Sigma_q)$  will be called *coherent* (with respect to the family  $\{H_q^s(M) | s \geq 0\}$ ) if its weighted Sobolev order and its Lyapunov exponent are related:

$$\mathcal{O}_q^H(\mathcal{D}^\pm) = |l_\mu^\pm(\mathcal{D}^\pm)|.$$

A finite dimensional space of invariant distributions will be called coherent (with respect to the family  $\{H_q^s(M) | s \geq 0\}$ ) if it has a basis of simple coherent elements.

By the definitions of coherence and regularity, the following result follows immediately from Theorem 4.35:

**Lemma 5.13.** *Let  $\mu$  be any  $G_t$ -invariant ergodic probability measure on any stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable holomorphic quadratic differentials. For any  $q \in \mathcal{R}_\mu$ , any coherent finite dimensional space of invariant distributions is regular (with respect to the family  $\{H_q^s(M) | s \geq 0\}$ ).*

For any  $q \in \mathcal{M}_\kappa^{(1)}$  and for any  $k \in \mathbb{N} \setminus \{0\}$ , let  $\mathcal{J}_q^k(M) \subset \mathcal{I}_q^k(M)$  be the space of horizontally invariant distributions defined as follows:

$$(5.62) \quad \mathcal{J}_q^k(M) := \bigoplus_{j=0}^{k-1} \mathcal{L}_{T_q}^j [\mathcal{I}_q^1(M)] .$$

**Corollary 5.14.** *For any  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic measure  $\mu$  on any stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable quadratic differentials, for  $\mu$ -almost all orientable quadratic differential  $q \in \mathcal{M}_\kappa^{(1)}$  and for any  $k \in \mathbb{N}$ , the space  $\mathcal{J}_q^k(M) \subset \mathcal{I}_q^k(M)$  is coherent, hence regular, with respect to the family  $\{H_q^s(M) | s \geq 0\}$  of weighted Sobolev spaces.*

*Proof.* It follows immediately from Theorem 4.29 for the case  $k = 0$  and from Theorem 4.29 and Lemma 4.32 in the general case.  $\square$

It was proved in Theorem 2.24 that the family  $\{H_q^s(M) | s \geq 0\}$  of weighted Sobolev spaces admits a family of smoothing operators. The existence of a smoothing family together with the regularity of the distributional obstructions are the key elements of the interpolation theory for solutions of the cohomological equation. We formalize below the basic construction of smoothing projectors onto the perpendicular of a regular subspace.

**Definition 5.15.** Let  $\{H^s | s \geq 0\}$  be a 1-parameter family of normed spaces such that the embeddings (5.59) hold. A *smoothing projection* of degree  $\sigma \in \overline{\mathbb{R}}^+$  relative to a subspace  $\mathcal{J}^\sigma \subset (H^\sigma)^*$  is a family  $\{P^\sigma(\tau) | \tau \in (0, 1]\}$  of linear operators such that the operator  $P^\sigma(\tau) : H^0 \rightarrow [\mathcal{J}^\sigma]^\perp \subset H^\sigma$  is bounded for all  $\tau \in (0, 1]$  and the following estimates hold. For any  $r$ ,

$s \in [0, \sigma]$  and for any  $\epsilon > 0$ , there exists a constant  $C_{r,s}^\sigma(\epsilon) > 0$  such that, for all  $u \in [\mathcal{J}^\sigma \cap (H^s)^*]^\perp \subset H^s$  and for all  $\tau \in (0, 1]$ ,

$$(5.63) \quad \begin{aligned} |P^\sigma(\tau)(u) - u|_r &\leq C_{r,s}^\sigma(\epsilon) |u|_s \tau^{s-r-\epsilon}, & \text{if } s > r; \\ |P^\sigma(\tau)(u)|_r &\leq C_{r,s}^\sigma(\epsilon) |u|_s \tau^{s-r-\epsilon}, & \text{if } s \leq r. \end{aligned}$$

A smoothing projection relative to the trivial subspace  $\mathcal{J}^\sigma = \{0\} \subset H^\sigma$  will be called a *smoothing* of degree  $\sigma \in \overline{\mathbb{R}}^+$  for the family  $\{H^s | s \geq 0\}$ .

The following result is a straightforward generalization of Theorem 2.24, which implies the existence of smoothings of any finite degree for the family  $\{H_q^s(M) | s \geq 0\}$  of weighted Sobolev spaces.

**Theorem 5.16.** *Let  $\{H^s | s \geq 0\}$  be a 1-parameter family of normed spaces such that the embeddings (5.59) hold. If the family  $\{H^s | s \geq 0\}$  has a smoothing of degree  $\sigma \in \mathbb{R}^+$  and the subspace  $\mathcal{J}^\sigma \subset (H^\sigma)^*$  is  $\sigma$ -regular, there exists a smoothing projection of degree  $\sigma \in \mathbb{R}^+$  relative to  $\mathcal{J}^\sigma$ .*

*Proof.* Let  $\{\mathcal{S}^\sigma(\tau) | \tau \in (0, 1]\}$  be a smoothing of degree  $\sigma > 0$  for the family  $\{H^s | s \geq 0\}$  and let  $\{\mathcal{D}_1, \dots, \mathcal{D}_J\}$  be a  $\sigma$ -regular basis for the subspace  $\mathcal{J}^\sigma \subset (H^\sigma)^*$ . By the Definition 5.11 of regularity, for any  $\tau \in (0, 1]$ , there exists a dual basis  $\{u_1(\tau), \dots, u_J(\tau)\} \subset H^\sigma$  such that the estimates (5.61) hold for all  $0 \leq r \leq \sigma$  and all  $\epsilon > 0$ . For any  $u \in H^0$ , we define

$$(5.64) \quad P^\sigma(\tau)(u) := \mathcal{S}^\sigma(\tau)(u) - \sum_{j=1}^J \mathcal{D}_j(\mathcal{S}^\sigma(\tau)(u)) u_j(\tau).$$

We claim that the family  $\{P^\sigma(\tau) | \tau \in (0, 1]\}$  is a smoothing projection (of degree  $\sigma > 0$ ) relative to the subspace  $\mathcal{J}^\sigma \subset (H^\sigma)^*$ . It follows immediately from the definition that  $P^\sigma(\tau)(u) \in [\mathcal{J}^\sigma]^\perp \subset H^\sigma$  for any  $u \in H^0$ , hence the operators  $P^\sigma(\tau) : H^0 \rightarrow [\mathcal{J}^\sigma]^\perp \subset H^\sigma$  are well-defined, linear and bounded. We claim that, for any  $s \in [0, \sigma]$ , for any  $j \in \{1, \dots, J\}$  and for any  $\epsilon > 0$ , there exists a constant  $C_j^s(\epsilon) > 0$  such that, for all vectors  $u \in [\mathcal{J}^\sigma \cap (H^s)^*]^\perp \subset H^s$ , the following estimate holds:

$$(5.65) \quad |\mathcal{D}_j(\mathcal{S}^\sigma(\tau)(u))| \leq C_j^s(\epsilon) |u|_s \tau^{s-\mathcal{O}^H(\mathcal{D}_j)-\epsilon}, \quad \text{for all } \tau \in (0, 1].$$

In fact, if the order  $\mathcal{O}^H(\mathcal{D}_j) \geq s$ , since  $\mathcal{D}_j \in (H^{r_j})^*$  for any  $r_j > \mathcal{O}^H(\mathcal{D}_j)$  and the family  $\{\mathcal{S}^\sigma(\tau) | \tau \in (0, 1]\}$  is a smoothing for  $\{H^s | s \geq 0\}$ , for any  $\epsilon > 0$  there exists a constant  $A_j^s(\epsilon) > 0$  such that for all  $u \in H^s$  and all  $\tau \in (0, 1]$ ,

$$(5.66) \quad |\mathcal{D}_j(\mathcal{S}^\sigma(\tau)(u))| \leq |\mathcal{D}_j|_{r_j}^* |\mathcal{S}^\sigma(\tau)(u)|_{r_j} \leq A_j^s(\epsilon) |u|_s \tau^{s-r_j-\epsilon/2}.$$

If  $\mathcal{O}^H(\mathcal{D}_j) < s$ , since  $\mathcal{D}_j \in (H^{r_j})^* \subset (H^s)^*$ , for any  $\mathcal{O}^H(\mathcal{D}_j) < r_j < s$  and the family  $\{\mathcal{S}^\sigma(\tau) | \tau \in (0, 1]\}$  is a smoothing for  $\{H^s | s \geq 0\}$ , there

exists  $B_j^s(\epsilon) > 0$  such that, for all  $u \in [\mathcal{J}^\sigma \cap (H^s)^*]^\perp$  and all  $\tau \in (0, 1]$ ,

$$(5.67) \quad \begin{aligned} |\mathcal{D}_j(\mathcal{S}^\sigma(\tau)(u))| &= |\mathcal{D}_j(\mathcal{S}^\sigma(\tau)(u) - u)| \\ &\leq |\mathcal{D}_j|_{r_j}^* |\mathcal{S}^\sigma(\tau)(u) - u|_{r_j} \leq B_j^s(\epsilon) |u|_s \tau^{s-r_j-\epsilon/2}. \end{aligned}$$

The estimate (5.65) follows immediately from estimates (5.66) and (5.67) by taking  $r_j \in (\mathcal{O}^H(\mathcal{D}_j), \mathcal{O}^H(\mathcal{D}_j) + \epsilon/2)$  for all  $j \in \{1, \dots, J\}$  in both cases.

By estimate (5.65) (just proved) and by the estimates on  $\{u_1(\tau), \dots, u_J(\tau)\}$  in formula (5.61), for any  $r \in [0, \sigma]$  and for any  $\epsilon > 0$ , there exists a constant  $C_{r,s}^{(1)}(\epsilon) > 0$  such that, for all  $u \in [\mathcal{J}^\sigma \cap (H^s)^*]^\perp$  and all  $\tau \in (0, 1]$ ,

$$(5.68) \quad \sum_{j=1}^J |\mathcal{D}_j(\mathcal{S}^\sigma(\tau)(u))| |u_j(\tau)|_r \leq C_{r,s}^{(1)}(\epsilon) |u|_s \tau^{s-r-\epsilon}.$$

Finally, the required estimates (5.63) for the family  $\{P^\sigma(\tau) | \tau \in (0, 1]\}$ , defined in (5.64), follow from (5.68), since  $\{\mathcal{S}^\sigma(\tau) | \tau \in (0, 1]\}$  is a smoothing (of degree  $\sigma > 0$ ) for the family  $\{H^s | s \geq 0\}$ .  $\square$

The smoothness of the solutions of the cohomological equation is best expressed with respect to the following uniform norms.

**Definition 5.17.** For any  $k \in \mathbb{N}$ , let  $B_q^k(M)$  be the space of all functions  $u \in H_q^k(M)$  such that  $S_q^i T_q^j u = T_q^i S_q^j u \in L^\infty(M)$  for all pairs of integers  $(i, j)$  such that  $0 \leq i + j \leq k$ . The space  $B_q^k(M)$  is endowed with the norm defined as follows: for any  $u \in B_q^k(M)$ ,

$$(5.69) \quad |u|_{k,\infty} := \left[ \sum_{i+j \leq k} |S_q^i T_q^j u|_\infty^2 \right]^{1/2} = \left[ \sum_{i+j \leq k} |T_q^i S_q^j u|_\infty^2 \right]^{1/2}.$$

For  $s \in [k, k+1)$ , let  $B_q^s(M) := B_q^k(M) \cap H_q^s(M)$  endowed with the norm defined as follows: for any  $u \in B_q^s(M)$ ,

$$(5.70) \quad |u|_{s,\infty} := (|u|_{k,\infty}^2 + |u|_s^2)^{1/2}.$$

**Theorem 5.18.** Let  $\mu$  be a  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic measure on a stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable quadratic differentials. For  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for any  $k \in \mathbb{N}$ , the space  $\mathcal{J}_q^k(M)$  is coherent, hence regular (with respect to the family  $\{H_q^s(M) | s \geq 0\}$ ) and, for any  $s > k$ , there exists a measurable function  $C_\kappa^{k,s} : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that the following holds. For any function  $f \in [\mathcal{J}_q^k(M)]^\perp \cap H_q^s(M)$  the Green

solution  $\mathcal{U}_q(f)$  of the cohomological equation  $S_q u = f$  belongs to the space  $B_q^{k-1}(M)$  and satisfies the estimates:

$$(5.71) \quad |\mathcal{U}_q(f)|_{k-1} \leq |\mathcal{U}_q(f)|_{k-1,\infty} \leq C_\kappa^{k,s}(q) |f|_s$$

*Proof.* For  $k = 1$  the statement reduces to Theorem 5.10 and Corollary 5.14. Since  $f \in [\mathcal{J}_q^2(M)]^\perp$  implies by definition that  $f$  and  $T_q f \in [\mathcal{J}_q^1(M)]^\perp$ , by Theorem 5.10, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for any  $s > k = 2$ , the cohomological equations  $S_q u = f$  and  $S_q u_T = T_q f$  have Green (zero average) solutions  $u(f)$  and  $u_T(f) \in L^\infty(M) \subset L_q^2(M)$  which satisfy the bounds,

$$(5.72) \quad \begin{aligned} |u(f)|_\infty &\leq C_\kappa^{s-1}(q) |f|_{s-1}, \\ |u_T(f)|_\infty &\leq C_\kappa^{s-1}(q) |f|_s. \end{aligned}$$

It follows that the distribution  $u_T(f) - T_q u(f) \in H_q^{-1}(M)$  is horizontally invariant. Let  $\{\mathcal{D}_1, \dots, \mathcal{D}_g\} \subset \mathcal{J}_q^1(M)$  be any regular basis such that

$$(5.73) \quad \mathcal{O}_q^H(\mathcal{D}_j) = 1 - \lambda_j^\mu, \quad \text{for all } j \in \{1, \dots, g\}.$$

It is no restrictive to assume that  $\mathcal{D}_1$  is the average. Since Green solutions have zero average by definition, there exists  $F_2(f), \dots, F_g(f) \in \mathbb{C}$  such that

$$(5.74) \quad u_T(f) - T_q u(f) = \sum_{j=2}^g F_j(f) \mathcal{D}_j.$$

It follows from the bounds (5.72) that the maps  $F_j : [\mathcal{J}_q^2(M)]^\perp \rightarrow \mathbb{C}$  are linear bounded functionals, for all  $j \in \{2, \dots, g\}$ , defined on the closed subspace  $[\mathcal{J}_q^2(M)]^\perp \subset H_q^s(M)$ . In fact, let  $\{u_1(\tau), \dots, u_g(\tau)\} \subset H_q^1(M)$  be any dual basis of the regular basis  $\{\mathcal{D}_1, \dots, \mathcal{D}_g\}$  as in Definition 5.11. By Theorem 5.10, for any  $s > 2$  and for any  $(\tau_2, \dots, \tau_g) \in (0, 1]^{g-1}$ ,

$$(5.75) \quad \begin{aligned} |F_j(f)| &= |\langle u_T(f), u_j(\tau_j) \rangle_q| + |\langle u(f), T_q u_j(\tau_j) \rangle_q| \\ &\leq C_\kappa^{s-1}(q) \{ |u_j(\tau_j)|_0 |f|_s + |u_j(\tau_j)|_1 |f|_{s-1} \}. \end{aligned}$$

We claim that, for each  $j \in \{2, \dots, g\}$ , the linear functional  $F_j$  extends to a horizontally invariant distribution  $\Phi_j \in H_q^{-s}(M)$  such that

$$(5.76) \quad \mathcal{O}_q^H(\Phi_j) \leq 1 + \lambda_j^\mu, \quad \text{for all } j \in \{2, \dots, g\}.$$

In fact, since by Corollary 5.14 the space  $\mathcal{J}_q^2(M)$  is regular, by Theorem 5.16 there exists a smoothing projection  $\{P_j^\sigma(\tau) | \tau \in (0, 1]\}$  of degree  $\sigma > 2$  relative to the subspace  $\mathcal{J}_q^2(M) \subset H_q^{-2}(M)$ . Hence by definition, for any  $r$ ,  $s \in [0, \sigma]$  and any  $\epsilon > 0$ , there exists a constant  $C_{r,s}^\sigma(\epsilon) > 0$  such that, for

all  $f \in [\mathcal{J}_q^2(M)]^\perp \cap H_q^s(M)$  and for all  $\tau \in (0, 1]$ ,

$$(5.77) \quad \begin{aligned} |P_J^\sigma(\tau)(f) - f|_r &\leq C_{r,s}^\sigma(\epsilon) |u|_s \tau^{s-r-\epsilon}, & \text{if } s > r; \\ |P_J^\sigma(\tau)(f)|_r &\leq C_{r,s}^\sigma(\epsilon) |f|_s \tau^{s-r-\epsilon}, & \text{if } s \leq r. \end{aligned}$$

Let  $(f_n)_{n \in \mathbb{N}}$  be the sequence of functions defined as follows:

$$(5.78) \quad f_n := P_J^\sigma(2^{-n})(f) \in [\mathcal{J}_q^2(M)]^\perp \cap H_q^\sigma(M), \quad \text{for all } n \in \mathbb{N}.$$

It follows from estimates (5.77) that, if  $s - 1 < 1 + \lambda_j^\mu < s_j < s$ , for any  $\epsilon > 0$ , there exists a constant  $C_{s_j,s}^\sigma(\epsilon) > 0$  such that

$$(5.79) \quad \begin{aligned} |f_{n+1} - f_n|_{s-1} &\leq C_{s_j,s}^\sigma(\epsilon) |f|_{s_j} 2^{-n(s_j-s+1-\epsilon)}; \\ |f_{n+1} - f_n|_s &\leq C_{s_j,s}^\sigma(\epsilon) |f|_{s_j} 2^{n(s-s_j+\epsilon)}. \end{aligned}$$

Let  $\{u_1^{(n)}, \dots, u_g^{(n)}\} \subset H_q^1(M)$  be the sequence of dual basis of the regular basis  $\{\mathcal{D}_1, \dots, \mathcal{D}_g\} \subset \mathcal{J}_q^1(M)$  defined as follows: for each  $j \in \{1, \dots, g\}$ ,

$$(5.80) \quad u_j^{(n)} := u_j(2^{-n}), \quad \text{for all } n \in \mathbb{N}.$$

By the estimate (5.61) in Definition (5.11) and by the identities (5.73), for any  $\epsilon > 0$  there exists a constant  $C(\epsilon) > 0$  such that, for all  $j \in \{1, \dots, g\}$ ,

$$(5.81) \quad \begin{aligned} |u_j^{(n)}|_0 &\leq C(\epsilon) 2^{-n(1-\lambda_j^\mu-\epsilon)}; \\ |u_j^{(n)}|_1 &\leq C(\epsilon) 2^{n(\lambda_j^\mu+\epsilon)}. \end{aligned}$$

For any  $s_j > 1 + \lambda_j$ , there exists  $s \in (2, 2 + \lambda_j^\mu)$  and  $\epsilon > 0$  such that

$$(5.82) \quad s_j + 1 - s - \lambda_j^\mu - 2\epsilon := \alpha_j > 0.$$

Hence by the estimate (5.61) and (5.75), for  $\mu$ -almost  $q \in \mathcal{M}_\kappa^{(1)}$ , there exists a constant  $C_q^j(\epsilon) := C_q(\epsilon, s, s_j, \alpha_j) > 0$  such that

$$(5.83) \quad |F_j(f_{n+1} - f_n)| \leq C_q^j(\epsilon) 2^{-\alpha_j n} |f|_{s_j}.$$

By (5.77) and (5.78) the sequence  $f_n \rightarrow f$  in  $H_q^s(M)$  for any  $s < \sigma$ . Since (5.75) holds for any  $s > 2$ , it follows from (5.83) that, for  $\mu$ -almost  $q \in \mathcal{M}_\kappa^{(1)}$ ,

$$(5.84) \quad |F_j(f)| \leq \frac{C_q^j(\epsilon)}{1 - 2^{-\alpha_j}} |f|_{s_j} + |P_J^\sigma(1)(f)|_\sigma.$$

Since the operator  $P_J^\sigma(1) : L_q^2(M) \rightarrow H_q^\sigma(M)$  is bounded, we conclude that for any  $s_j > 1 + \lambda_j^\mu$  there exists a continuous extension  $\Phi_j^{s_j} \in H_q^{-s_j}(M)$  of the linear functional  $F_j : [\mathcal{J}_q^2(M)]^\perp \cap H_q^s(M) \rightarrow \mathbb{C}$ . By construction, for any  $r_j, s_j > 1 + \lambda_j^\mu$ , the extensions  $\Phi_j^{r_j} = \Phi_j^{s_j} \pmod{\mathcal{J}_q^2(M)}$ . Since  $\mathcal{J}_q^2(M)$  is finite dimensional, for each  $j \in \{2, \dots, g\}$  there exists a distribution  $\Phi_j \in H_q^{-2}(M)$ , which extends the linear functional  $F_j$ , such that



the Sobolev order  $\mathcal{O}_q^H(\Phi_j) \leq 1 + \lambda_j^\mu$  as claimed in (5.76). Finally, we prove that, by construction, the distributions  $\Phi_2, \dots, \Phi_g \in H_q^2(M)$  are horizontally invariant. In fact, for any  $s > 2$  and any  $v \in \mathcal{H}_q^{s+1}(M)$ , the function  $f_v := S_q v \in [\mathcal{J}_q^2(M)]^\perp \cap H_q^s(M)$ , hence, by Theorem 5.10, the cohomological equations  $S_q u = f_v$  and  $S_q u_T = T_q f_v$  have unique Green solutions  $u(f_v)$  and  $u_T(f_v) \in L^\infty(M)$  respectively. By the ergodicity of the horizontal foliation, since  $v$  and  $T_q v \in L_q^2(M)$  are also zero-average solutions, the identities  $u(f_v) = v$  and  $u_T(f_v) = T_q v$  hold. It follows that  $u_T(f_v) - T_q u(f_v) = 0$ , hence by (5.74), for all  $j \in \{2, \dots, g\}$ ,

$$(5.85) \quad \Phi_j(S_q v) = F_j(S_q v) = 0, \quad \text{for all } v \in H_q^{s+1}(M),$$

thus  $\{\Phi_2, \dots, \Phi_g\} \subset \mathcal{J}_q^s(M)$  and the claim is completely proved.

For  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for all integers  $k \geq 2$ , let  $\hat{\mathcal{J}}_q^k(M) \subset \mathcal{J}_q^k(M)$  be the subspaces defined as follows:

$$(5.86) \quad \hat{\mathcal{J}}_q^k(M) := \bigcup_{h=0}^{k-2} \mathcal{L}_{T_q}^h \left[ \mathcal{J}_q^2(M) \oplus \bigoplus_{j=2}^g \mathbb{C} \cdot \Phi_j \right].$$

It follows from the above construction that the following holds: for any integer  $k \geq 2$ , for any  $s > k$  and for any function  $f \in H_q^s(M) \cap [\hat{\mathcal{J}}_q^k(M)]^\perp$ , the Green solution  $\mathcal{U}_q(f) \in B_q^{k-1}(M)$  and satisfies the required estimate (5.71). In fact, for  $k = 2$  the statement follows by the construction of the system of invariant distributions  $\{\Phi_2, \dots, \Phi_g\} \subset H_q^{-2}(M)$ . For  $k \geq 3$ , the statement can be proved by induction. In fact, by the induction hypothesis, for any  $s > k$  and any  $f \in H_q^s(M) \cap [\hat{\mathcal{J}}_q^k(M)]^\perp$ , the cohomological equations  $S_q u = f$  has a unique solution  $u \in B_q^{k-2}(M)$  such that

$$(5.87) \quad |u|_{k-2, \infty} \leq C_\kappa^{k-1, s-1}(q) |f|_{s-1}.$$

In addition, the function  $u_T := T_q u \in H_q^{k-3}(M)$  is the unique solution of the cohomological equation  $S_q u_T = T_q f$ . Since  $T_q f \in H_q^{s-1}(M) \cap [\hat{\mathcal{J}}_q^{k-1}(M)]^\perp$ , by the induction hypothesis, the following estimate holds:

$$(5.88) \quad |T_q u|_{k-2, \infty} \leq C_\kappa^{k-1, s-1}(q) |T_q f|_{s-1} \leq C_\kappa^{k-1, s-1}(q) |f|_s.$$

Finally, by the Sobolev embedding theorem, there exists a continuous function  $\tilde{C}_\kappa : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that the following estimate holds:

$$(5.89) \quad \left[ \sum_{i=1}^{k-1} |S_q^i u|_\infty^2 \right]^{1/2} = \left[ \sum_{i=0}^{k-2} |S_q^i f|_\infty^2 \right]^{1/2} \leq \tilde{C}_\kappa(q) |f|_s.$$

The required estimate follows from (5.87), (5.88) and (5.89).

It remains to be proven that, for  $\mu$ -almost  $q \in \mathcal{M}_\kappa^{(1)}$  and all integers  $k \geq 2$ , the space  $\mathcal{J}_q^k(M)$  is coherent. From the above argument it follows that

$$(5.90) \quad \mathcal{J}_q(M) = \bigoplus_{k=2}^{+\infty} \hat{\mathcal{J}}_q^k(M).$$

In fact, if  $f \in H_q^\infty(M)$  is such that  $f \in [\hat{\mathcal{J}}_q^k(M)]^\perp$  for all  $k \in \mathbb{N}$ , there exists  $u \in H_q^\infty(M)$  such that  $S_q u = f$ , hence  $f \in \mathcal{J}_q(M)^\perp$ . The identity (5.90) immediately implies that

$$(5.91) \quad \mathcal{J}_q^k(M) = \hat{\mathcal{J}}_q^k(M), \quad \text{for all integers } k \geq 2.$$

The identities (5.91) in turn imply that the system  $\{\Phi_2, \dots, \Phi_g\}$  constructed above is linearly independent over  $\mathcal{J}_q^2(M)$ , in particular

$$(5.92) \quad \dim_{\mathbb{C}} \mathcal{J}_q^2(M) = \dim_{\mathbb{C}} \hat{\mathcal{J}}_q^2(M) = 3g - 2.$$

In fact, by the above construction the following identity holds:

$$(5.93) \quad \mathcal{J}_q(M) = \mathcal{J}_q^1(M) + \bigoplus_{j=2}^g \mathbb{C} \cdot \Phi_j + \mathcal{L}_{T_q}[\mathcal{J}_q(M)].$$

Let  $\mathcal{D}_q : \mathcal{B}_q(M) \rightarrow \mathcal{J}_q(M)$  the isomorphism between the space  $\mathcal{B}_q(M)$  of horizontally basic currents and the space of horizontally invariant distributions defined in (3.34). Let  $\delta_q : \mathcal{B}_q(M) \rightarrow \mathcal{B}_q(M)$  be the differential map on the space of basic currents introduced in (3.61). From (5.93) it follows immediately that

$$(5.94) \quad \mathcal{B}_q(M) = \mathcal{B}_q^1(M) + \bigoplus_{j=2}^g \mathbb{C} \cdot \mathcal{D}_q^{-1}(\Phi_j) + \delta_q[\mathcal{B}_q(M)].$$

By the structure theorem for (real) basic currents (Theorem 3.21), the cohomology map  $j_q : \mathcal{B}_q(M) \rightarrow H^1(M, \mathbb{R})$  vanishes on the space  $\delta_q[\mathcal{B}_q(M)]$ . On the other hand, by Corollary 3.20, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , the cohomology map on  $\mathcal{B}_q(M)$  has rank of codimension 1 in the homology  $H^1(M, \mathbb{R})$ , hence of dimension  $2g - 1$ . Since  $\mathcal{B}_q^1(M)$  has dimension  $g$  and the map  $\mathcal{D}_q : \mathcal{B}_q(M) \rightarrow \mathcal{J}_q(M)$  is an isomorphism, it follows that the system  $\{\Phi_2, \dots, \Phi_g\}$  is linearly independent over  $\mathcal{J}_q^2(M)$  and (5.92) holds.

We claim that, for  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$ , the space  $\hat{\mathcal{J}}_q^2(M) = \mathcal{J}_q^2(M)$  is coherent. For any Lyapunov exponent  $l < 0$  of the cocycle  $\Phi_t^2|_{\mathcal{J}_q^2(M)}$ , let  $E_q^2(l) \subset \mathcal{J}_q^2(M)$  denote the corresponding Oseledec subspace and let  $F_q^2(l) \subset E_q^2(l)$  be the subspace of coherent distributions. Let

$$l_1 < \dots < l_d$$

be the distinct Lyapunov exponents of the cocycle  $\Phi_t^2|_{\mathcal{J}_q^2(M)}$  on the the Oseledec complement of the subspace  $\mathcal{J}_q^2(M)$ . Since  $\mathcal{J}_q^2(M)$  is coherent and

$$\mathcal{J}_q^2(M) = \hat{\mathcal{J}}_q^2(M) = \mathcal{J}_q^2(M) \oplus \bigoplus_{i=1}^d E_q^2(l_i),$$

it is sufficient to prove the identities:

$$(5.95) \quad F_q^2(l_i) = E_q^2(l_i), \quad \text{for all } i \in \{1, \dots, d\}.$$

By Theorem 4.30 and Lemma 4.32, the Lyapunov spectrum of  $\{\Phi_t^2|_{\mathcal{J}_q^2(M)}\}$  is the ordered set

$$0 > \lambda_2^\mu - 1 \geq \dots \geq \lambda_g^\mu - 1 \geq \lambda_2^\mu - 2 \geq \dots \geq \lambda_g^\mu - 2.$$

Hence, by the description of the Lyapunov spectrum of  $\{\Phi_t^2|_{\mathcal{J}_q^2(M)}\}$  given in Corollary 4.33, the set

$$\{l_1, \dots, l_d\} = \{-1 - \lambda_2^\mu, \dots, -1 - \lambda_g^\mu\}.$$

For any  $s \geq 0$ , let  $\mathcal{J}_q(s) \subset E_q^2(l_1) \oplus \dots \oplus E_q^2(l_d)$  be the subset of horizontally invariant distributions of Sobolev order less or equal to  $s \geq 0$ . It follows from Lemma 4.26 that the following inclusions hold:

$$(5.96) \quad \mathcal{J}_q(|l_i|) \subset \bigoplus_{j \leq i} E_q^2(l_j), \quad \text{for all } i \in \{1, \dots, d\}.$$

By the estimate (5.76) on Sobolev orders of the distributions in the system  $\{\Phi_2, \dots, \Phi_g\}$ , the following lower bounds hold:

$$(5.97) \quad \dim_{\mathbb{C}} I_q(|l_i|) \geq \sum_{j=1}^i \dim_{\mathbb{C}} E_q^2(l_j), \quad \text{for all } i \in \{1, \dots, d\}.$$

It follows from (5.96) and (5.97) that the inclusions in (5.96) are in fact identities, for all  $i \in \{1, \dots, d\}$ , and by Lemma 4.26 the claim (5.95) holds.

We have thus proved that the space  $\mathcal{J}_q^2(M) = \hat{\mathcal{J}}_q^2(M)$  is coherent. It follows from definition (5.86) and Lemma 4.32 that the space  $\hat{\mathcal{J}}_q^k(M)$  is coherent for any integer  $k \geq 3$ . Since by (5.91) the identity  $\mathcal{J}_q^k(M) = \hat{\mathcal{J}}_q^k(M)$  holds for all  $k \geq 2$ , the space  $\mathcal{J}_q^k(M)$  is coherent and the proof is complete.  $\square$

**Theorem 5.19.** *Let  $\mu$  be a  $SO(2, \mathbb{R})$ -absolutely continuous, KZ-hyperbolic measure on a stratum  $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$  of orientable quadratic differentials. For  $\mu$ -almost all  $q \in \mathcal{M}_\kappa^{(1)}$  and for any  $s \in \mathbb{R}^+$ , the space  $\mathcal{J}_q^s(M)$  is coherent, hence regular (with respect to the family  $\{H_q^s(M) | s \geq 0\}$ ) and, for any  $0 < r < s - 1$  there exists a measurable function  $C_\kappa^{r,s} : \mathcal{M}_\kappa^{(1)} \rightarrow \mathbb{R}^+$  such that the following holds. For any function  $f \in [\mathcal{J}_q^s(M)]^\perp \cap H_q^s(M)$  the*

Green solution  $\mathcal{U}_q(f)$  of the cohomological equation  $S_q u = f$  belongs to the space  $H_q^r(M)$  and satisfies the estimates:

$$(5.98) \quad |\mathcal{U}_q(f)|_r \leq C_{\kappa}^{r,s}(q) |f|_s$$

*Proof.* By Theorem 5.18, the subspace  $\mathcal{J}_q^k(M)$  is coherent for  $\mu$ -almost all  $q \in \mathcal{M}_{\kappa}^{(1)}$  and for any  $k \in \mathbb{N}$ . Since for any  $s < k$  the sub-bundle  $\mathcal{J}_{\kappa,+}^s(M) \subset \mathcal{J}_{\kappa,+}^k(M)$  is  $\{\Phi_t^k\}$ -invariant, it follows that  $\mathcal{J}_q^s(M)$  is coherent, hence regular by Lemma 5.13, for  $\mu$ -almost all  $q \in \mathcal{M}_{\kappa}^{(1)}$ .

By Theorem 5.16, for  $\mu$ -almost all  $q \in \mathcal{M}_{\kappa}^{(1)}$  there exists a smoothing projection  $\{P^\sigma(\tau) | \tau \in (0, 1]\}$  of any given degree  $\sigma > 0$  relative to the subspace  $\mathcal{J}_q^\sigma(M)$ . Let  $s > 1$  and let  $\sigma \in \mathbb{R}^+$  and  $k \in \mathbb{N}$  be such that  $\sigma > k + 1 \geq s \geq \sigma - 1 > k$ . Let  $f \in [\mathcal{J}_q^s(M)]^\perp \cap H_q^s(M)$ . By Definition 5.15 of a smoothing projection, for any  $f \in [\mathcal{J}_q^s(M)]^\perp \cap H_q^s(M)$ , the function  $P^\sigma(\tau)(f) \in [\mathcal{J}_q^\sigma(M)]^\perp \cap H_q^\sigma(M)$  and satisfies the Sobolev bounds (5.63). By Theorem 5.18 the cohomological equation  $S_q u = P^\sigma(\tau)(f)$  has a (unique) Green solution  $u(\tau) \in H_q^{k-1}(M)$  and there exists a measurable function  $C_\kappa : \mathcal{M}_{\kappa}^{(1)} \rightarrow \mathbb{R}^+$  such that, for all  $\tau \in (0, 1]$ ,

$$(5.99) \quad \begin{aligned} |u(\tau) - u(\tau/2)|_k &\leq C_\kappa(q) |P^\sigma(\tau)(f) - P^\sigma(\tau/2)(f)|_\sigma, \\ |u(\tau) - u(\tau/2)|_{k-1} &\leq C_\kappa(q) |P^\sigma(\tau)(f) - P^\sigma(\tau/2)(f)|_{\sigma-1}. \end{aligned}$$

By the interpolation inequality proved in Lemma 2.10, for any  $r \in [k-1, k]$  there exists  $C_{k,r} > 0$  such that, for all  $\tau \in (0, 1]$ ,

$$(5.100) \quad |u(\tau) - u(\tau/2)|_r \leq C_{k,r} |u(\tau) - u(\tau/2)|_{k-1}^{k-r} |u(\tau) - u(\tau/2)|_k^{r-k+1}.$$

By the bounds (5.63), it follows from (5.99) and (5.99) that, for any  $\epsilon > 0$  there exists  $C_{r,s}^\sigma(\epsilon) > 0$  such that

$$(5.101) \quad |u(\tau) - u(\tau/2)|_r \leq C_{r,s}^\sigma(\epsilon) |f|_s \tau^{(k-r)(s-\sigma+1-\epsilon)} \tau^{(r-k+1)(s-\sigma-\epsilon)}.$$

Since  $r < s - 1$ , it is possible to choose  $\sigma \in \mathbb{R}^+$ ,  $\epsilon > 0$  and  $k \in \mathbb{N}$  so that

$$\alpha = (k-r)(s-\sigma+1-\epsilon) + (r-k+1)(s-\sigma-\epsilon) > 0.$$

It follows then from the bound (5.101), that the sequence  $\{u(2^{-n}) | n \in \mathbb{N}\}$  is Cauchy in  $H_q^r(M)$ , hence it converges to a function  $u \in H_q^r(M)$  of zero average. Since  $P^\sigma(\tau)(f) \rightarrow f$  in  $H_q^s(M)$  as  $\tau \rightarrow 0^+$ , it follows that the function  $u \in H_q^r(M)$  is the unique zero-average (Green) solution of the cohomological equation  $S_q u = f$ . The required Sobolev bound (5.98) also follows from (5.101). In fact, by the interpolation inequality (Lemma 2.10), by Theorem 5.18 and by the bounds (5.63) for the smoothing projection, there exists a measurable function  $C'_\kappa : \mathcal{M}_{\kappa}^{(1)} \rightarrow \mathbb{R}^+$  such that,

$$(5.102) \quad |u(1)|_r \leq C'_\kappa(q) |f|_s.$$

The bound (5.98) can then be derived from (5.101) and (5.102).  $\square$

## REFERENCES

- [Ath06] J. Athreya, *Quantitative Recurrence and Large Deviations for Teichmüller geodesic flow*, Ph. D. thesis, University of Chicago, 2006.
- [AV05] A. Avila and M. Viana, *Simplicity of Lyapunov spectra: Proof of the Zorich-Kontsevich conjecture*, preprint on arXiv:math.DS/0508508, 2005.
- [Ber74] L. Bers, *Spaces of degenerating Riemann surfaces*, Discontinuous groups and Riemann surfaces, Proc. Conf. Univ. of Maryland, College Park, Md, 1973 (Princeton, NJ), Annals of Mathematical Studies, vol. 79, Princeton University Press, 1974, pp. 43–55.
- [BP05] L. Barreira and Y. Pesin, *Smooth ergodic theory and non-uniformly hyperbolic dynamics*, Handbook of Dynamical Systems (B. Hasselblatt and A. Katok, eds.), vol. 1B, Elsevier, 2005.
- [CEG84] P. Collet, H. Epstein, and G. Gallavotti, *Perturbations of geodesic flows on surfaces of constant negative curvature and their mixing properties*, Comm. Math. Phys. **95** (1984), 61–112.
- [dLMM86] R. de la Llave, J. M. Marco, and R. Moriyón, *Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation*, Ann. of Math. **123** (1986), 537–611.
- [FF03] L. Flaminio and G. Forni, *Invariant distributions and time averages for horocycle flows*, Duke Math. J. **119** (2003), no. 3, 465–526.
- [FF06] ———, *Equidistribution of nilflows and applications to theta sums*, Ergodic Theory Dynam. Systems **26** (2006), no. 02, 409–433.
- [FF07] ———, *On the cohomological equation for nilflows*, Journal of Modern Dynamics **1** (2007), no. 1, 37–60.
- [FK92] H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, New York, NY, 1992, second edition.
- [For97] G. Forni, *Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus*, Ann. of Math. (2) **146** (1997), no. 2, 295–344.
- [For02] ———, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Ann. of Math. (2) **155** (2002), no. 1, 1–103.
- [For05] ———, *On the Lyapunov exponents of the Kontsevich-Zorich cocycle*, Handbook of Dynamical Systems (B. Hasselblatt and A. Katok, eds.), vol. 1B, Elsevier, 2005.
- [GH55] W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, AMS Colloquium Publications, vol. 36, AMS, Providence, R.I., 1955.
- [GK80] V. Guillemin and Kazhdan, *Some inverse spectral results for negatively curved 2-manifolds*, Topology **19** (1980), 301–312.
- [Her83] M. R. Herman, *Sur les courbes invariantes par les difféomorphismes de l’anneau, vol. 1*, Asterisque, vol. 103-104, Société Mathématique de France, Paris, France, 1983.
- [HS05] P. Hubert and T. Schmidt, *Affine diffeomorphisms and the Veech Dichotomy*, Handbook of Dynamical Systems (B. Hasselblatt and A. Katok, eds.), vol. 1B, Elsevier, 2005.

- [Kon97] M. Kontsevich, *Lyapunov exponents and Hodge theory*, The mathematical beauty of physics, Saclay, 1996 (River Edge, NJ), Adv. Ser. Math. Phys., vol. 24, World Scientific, 1997, pp. 318–332.
- [KZ03] M. Kontsevich and A. Zorich, *Connected components of the moduli space of abelian differentials with prescribed singularities*, Inv. Math. **153** (2003), 631–678.
- [Lan03] E. Lanneau, *Connected components of the moduli spaces of quadratic differentials*, Thèse de Doctorat, Université de Rennes, 2003.
- [Liv71] A. Livsic, *Homology properties of U-systems*, Math. Notes USSR Acad. Sci. **10** (1971), 758–763, (in Russian).
- [LM68] J. L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications, vol. 1*, Dunod, Paris, 1968.
- [Mas82] H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. **115** (1982), 168–200.
- [Mas93] ———, *Logarithmic law for geodesics in moduli space*, Mapping Class Groups and Moduli Spaces of Riemann surfaces, Proceedings, Göttingen June-August 1991 (Providence, RI), Contemporary Mathematics, vol. 150, AMS, 1993, pp. 229–245.
- [Mas05] ———, *Ergodic Theory of Translation Surfaces*, Handbook of Dynamical Systems (B. Hasselblatt and A. Katok, eds.), vol. 1B, Elsevier, 2005.
- [MMY03] S. Marmi, P. Moussa, and J.-C. Yoccoz, *On the cohomological equation for interval exchange maps*, C. R. Math. Acad. Sci. Paris **336** (2003), 941–948.
- [MMY05] S. Marmi, P. Moussa, and J.-C. Yoccoz, *The cohomological equation for Roth type interval exchange maps*, J. Amer. Math. Soc. **18** (2005), 823–872.
- [MT05] H. Masur and S. Tabachnikov, *Rational billiards and flat structures*, Handbook of Dynamical Systems (B. Hasselblatt and A. Katok, eds.), vol. 1A, Elsevier, 2005, pp. 1015–1089.
- [Nag88] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, John Wiley & Sons, Inc., New York, NY, 1988.
- [Nel59] E. Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), 572–615.
- [Vee82] W. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. **115** (1982), 201–242.
- [Vee86] ———, *The Teichmüller geodesic flow*, Ann. of Math. **124** (1986), 441–530.
- [Vee90] ———, *Moduli spaces of quadratic differentials*, J. D’Analyse Math. **55** (1990), 117–171.
- [Zor94] A. Zorich, *Asymptotic flag of an orientable measured foliation on a surface*, pp. 479–498, World Scientific, 1994.
- [Zor96] ———, *Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents*, Ann. Inst. Fourier (Grenoble) **46** (1996), 325–370.
- [Zor97] ———, *Deviation for interval exchange transformations*, Ergodic Th. and Dynam. Syst. **17** (1997), 1477–1499.
- [Zor99] ———, *How Do the Leaves of a Closed 1-form Wind around a Surface?*, Pseudoperiodic Topology (Providence, RI) (M. Kontsevich V. I. Arnol’d and A. Zorich, eds.), Amer. Math. Soc. Transl. (2), vol. 197, AMS, 1999, pp. 135–178.

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